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Reduction of a Hamilton – Jacobi Equation for Nonholonomic Systems

Oğul Esen[1*], Victor M. Jiménez[2], Manuel de León[3***], and Cristina Sardón[2****]**

[1]*Department of Mathematics, Gebze Technical University, Gebze 41400, Kocaeli, Turkey*

[2]*ICMAT, Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera 13–144, 28049, Spain*

[3]*ICMAT, C/ Nicolás Cabrera 13–15, Campus Cantoblanco, UAM 28049 Madrid, Spain; Real Academia de Ciencias Exactas, Fisicas y Naturales, C/de Valverde 22, 28004 Madrid, Spain*

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Abstract—We discuss, in all generality, the reduction of a Hamilton – Jacobi theory for systems subject to nonholonomic constraints and invariant under the action of a group of symmetries. We consider nonholonomic systems subject to both linear and nonlinear constraints and with different positioning of such constraints with respect to the symmetries.

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This work is dedicated to the memory of Sergey A. Chaplygin (1869–1942)

1. INTRODUCTION

In this paper we aim at providing a general reduction of a Hamilton – Jacobi formulation for nonholonomic mechanical systems subject to nonlinear constraints and invariant under a group of symmetries. The reduction of a Hamilton – Jacobi theory for nonholonomic systems is considered of remarkable interest given the number of wheeled dynamical systems subject to this kind of constraints [1, 2], which appear in engineering problems, theoretical physics and mechanics. There is a difference between constrained systems and nonholonomic systems. Nonholonomic systems are not Hamiltonian, in the sense that the phase space is just the constraint submanifold and not the cotangent bundle of the configuration manifold. The geometric tool to study the dynamics is an almost Poisson bracket.

Many frameworks have been proposed to solve the dynamics of linear/nonlinearly nonholonomic systems [3–6], but the Hamilton – Jacobi theory for nonholonomic systems had not been effective, or it was very restrictive because it tried to adapt the usual proof of Hamilton – Jacobi equations without constraints using Hamilton's principle. Usually, the results are only valid when the solutions of the nonholonomic problem are also the solutions of the corresponding constrained

*E-mail: oesen@gtu.edu.tr
**E-mail: victor.jimenez@icmat.es
***E-mail: mdeleon@icmat.es
****E-mail: cristinasardon@icmat.es

variational problem [7, 8]. M. de León et al. proposed an alternative approach based on geometrical properties [9], proving that nonholonomic mechanics is retrieved by projecting the unconstrained dynamics from the classical framework of the Hamilton – Jacobi formalism [10, 11]. This approach completes and engages all the partial results that had been resolved until that moment. We want to mention the seminal papers by J. Koiller [6] and A. M. Vershik and L. D. Faddev [12], as well as the treatment of nonholonomic mechanical systems with symmetries in A. Bloch et al. [3] that we use in the present paper.

The reduction of the geometric Hamilton – Jacobi theory for unconstrained systems was recently achieved through coisotropic reduction [11]. On the other hand, a first more general and effective reduction of a Hamilton – Jacobi theory for nonholonomic constraints was introduced in [10]. There, it is presented for the particular case in which the configuration space is a fibration over another manifold, and the constraints are given by the horizontal subspaces of a connection on this fibration [13]. In this case, the original nonholonomic system is equivalent to another one whose configuration manifold is the base of the fibration and, in addition, it is subject to an external force [14]. These are known as Chaplygin systems [15, 16]. Nonetheless, none of these works have been exhaustive showing the reduction of a nonholonomic system in the most general cases, nor considering the possibility of nonlinear constraints, although several systems possess nonlinear nonholonomic constraints [17–20].

In this paper, we aim at generalizing the reduction (by a group of symmetries) of a Hamilton – Jacobi theory for nonholonomic systems with nonlinear constraints, and as a particular example, the Chaplygin system is retrieved. Here, we will make a difference among different types of these systems depending on the relative positioning between the constraint functions and the symmetries. We also consider the case when the constraints (linear or not) are bracket generating or completely nonholonomic.

The organization of the paper is as follows: In Section 2 we recall the fundamentals of nonholonomic mechanics both in the Lagrangian and the Hamiltonian frameworks, we present geometric Hamilton – Jacobi theory for systems with external forces with linear and nonlinear constraints both in the Lagrangian and the Hamiltonian picture. Here we have included an extension of the result by M. de León et al [10, 21] and T. Ohsawa et al [22] to the case of nonlinear constraints. In Section 3 we introduce geometric Hamilton – Jacobi theory for the reduced nonholonomic systems with symmetries both in Lagrangian and Hamiltonian frameworks. Section 4 is devoted to the theory of reduction of nonholonomic systems according to the positioning of the symmetries with respect to the constraint functions. We display exhaustively geometric Hamilton – Jacobi theory for the pure kinematic and horizontal cases both for Lagrangian and Hamiltonian pictures.

For simplicity, we hereafter assume all mathematical objects to be real, smooth, and globally defined. This will permit us to omit several minor technical problems so as to highlight the main aspects of our results. We work on an n-dimensional configuration manifold Q and we choose coordinates as (q^i). We consider its tangent and cotangent bundles TQ and T^*Q with canonical projections τ_Q, and π_Q, respectively. In bundle coordinates, the tangent space is locally coordinated by $(q^i, \dot{q}^i)$ and the phase space T^*Q by the Darboux coordinates (q^i, p_i). The phase space T^*Q is equipped with a canonical one-form θ_Q, called the Liouville – Euler one-form. The canonical symplectic two-form ω_Q on T^*Q is minus of the exterior derivative of θ_Q [23–26]. In coordinates, θ_Q manifests its semibasic character, whereas the symplectic two-form is a constant skew symmetric two-form. They take the form:

$$\theta_Q = p_i dq^i, \qquad \omega_Q = dq^i \wedge dp_i.$$

2. HAMILTON – JACOBI THEORY FOR NONHOLONOMIC MECHANICS

2.1. Constrained Mechanics

A Lagrangian function L is a real valued function on the tangent bundle TQ of a manifold Q. In local coordinates $(q^i, \dot{q}^i)$, the Euler – Lagrange equations, generated by L, are computed to be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad \forall i = 1, \ldots, n. \tag{2.1}$$

Define the following $(1,1)$-tensor field:

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i \tag{2.2}$$

on the base manifold Q. In the literature, S is called the vertical endomorphism and the Poincaré – Cartan one-form is retrieved as

$$\alpha_L = S^*(dL) = \frac{\partial L}{\partial \dot{q}^i} dq^i. \tag{2.3}$$

The Poincaré – Cartan two-form ω_L on TQ is defined to be $\omega_L = -d\alpha_L$. If the Lagrangian is regular, then the Poincaré – Cartan two-form ω_L turns out to be a symplectic two-form. Then the pair (TQ, ω_L) becomes a symplectic manifold. For regular Lagrangians, the Legendre transformation

$$\mathbb{F}L : TQ \longrightarrow T^*Q : \left(q^i, \dot{q}^i\right) \mapsto \left(q^i, \frac{\partial L}{\partial \dot{q}^i}\right)$$

turns out to be a local diffeomorphism and the pullback of the canonical symplectic two-form ω_Q on T^*Q back to TQ by means of $\mathbb{F}L$ is precisely the Poincaré – Cartan two-form ω_L on TQ. In the present work, we will assume that all the Lagrangian functions L are regular. The energy function associated with a Lagrangian L is defined to be

$$E_L = \Delta(L) - L = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q, \dot{q}). \tag{2.4}$$

Here, Δ is the Liouville vector field on TQ which can be exhibited in local coordinates as $\Delta = \dot{q}^i(\partial/\partial \dot{q}^i)$. In terms of the Poincaré – Cartan two-form and the energy function, we can recast the Euler – Lagrange equations (2.1) as

$$\iota_{X_L} \omega_L - dE_L = 0. \tag{2.5}$$

A solution X_L of (2.5) is a second-order differential equation. This implies that the integral curves of X_L are tangent lifts of their projections to the configuration manifold Q. These projections are called the solutions of X_L and are just the solutions of the Euler – Lagrange equations [23].

Consider a nonholonomic Lagrangian system consisting of a regular Lagrangian function L on TQ and a submanifold $N \subseteq TQ$ of codimension k in TQ defining the nonholonomic constraints of the system. Then N may be locally described in terms of independent constraint functions $\{\psi^a\}$ in the following way:

$$N = \left\{(q^i, \dot{q}^j) \in TQ : \psi^a(q, \dot{q}) = 0\right\}. \tag{2.6}$$

We will always assume that $\tau_Q(N) = Q$ or, equivalently, the constraints are purely kinematic. According to the d'Alembert – Chetaev principle, the equation of motion of the constrained Lagrangian system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \frac{\partial \psi^a}{\partial \dot{q}^i}, \quad \psi^a(q, \dot{q}) = 0, \tag{2.7}$$

where λ_a are the Lagrange multipliers to be determined (see, for instance, [18, 27]).

A constraint submanifold N defines a distribution F_L on TQ along N whose annihilator F_L^o plays the role of "reaction forces" on TQ[1)]. These forces are one-forms on TQ along the submanifold N, and can be understood by taking their values in the subbundle

$$F_L^o = S^*(TN^o) = \left\langle \frac{\partial \psi^a}{\partial \dot{q}^i} dq^i \right\rangle \tag{2.8}$$

[1)] Notice that any distribution on a manifold M along a submanifold $\mathscr{S} \subseteq M$ can be naturally depicted as a generalized vector subbundle of the tangent bundle TM over $\mathscr{S}$. Throughout this paper we will use both terminologies.

of $T^*_N TQ$, where S is the vertical endomorphism, i. e., the $(1,1)$-tensor field, given in (2.2) on the base manifold TQ. Notice that the codistribution F^o_L is generated by semibasic one-forms. Thus, the equation of motion associated to the constrained Lagrangian system can be written as [18]:

$$\iota_{\Gamma_{L,N}}\omega_L - dE_L \in S^*(TN^o), \quad \Gamma_{L,N}|_N \in TN, \tag{2.9}$$

where ω_L is the symplectic two-form on TQ, and E_L is the energy of the Lagrangian L. It is remarkable that any solution $\Gamma_{L,N}$ of Eq. (2.9) is a SODE, since the one-forms in F^o_L are semibasic. Nonetheless, existence and uniqueness of a vector field $\Gamma_{L,N}$ satisfying Eq. (2.9) cannot be guaranteed. The system has a solution if and only if the value of the unconstrained vector field Γ_L falls into the sum $TN + F^{\perp}_L$ at every point in N. Here, $F^{\perp}_L$ denotes the symplectic complement of the distribution F_L with respect to the symplectic two-form ω_L. To guarantee existence and uniqueness of $\Gamma_{L,N}$, we will impose two conditions:

(i) Admissibility condition: for each $x \in N$

$$\dim (T_x N)^o = \dim \left(F^o_L\right)_x . \tag{2.10}$$

(ii) Compatibility condition: for each $x \in N$,

$$T_x N \cap (F^{\perp}_L)_x = \{0\}. \tag{2.11}$$

The admissibility condition simply implies that the restriction of the vertical endomorphism $S^* : TN^o \to F^o_L$ is an isomorphism or, equivalently, the family of one-forms $\{(\partial\psi^a/\partial\dot{q}^i)dq^i\}$ is linearly independent. To understand the compatibility condition, let us define the following matrix:

$$\mathscr{C} = \left(\mathscr{C}^{ab}\right) = \left(Z^a\left(\psi^b\right)\right),$$

where each Z^b is the (local) vector field on TQ such that $S^*\left(d\psi^b\right)$ is the image of Z^b by the musical isomorphism $\omega^{\flat}_L$ induced from ω_L, i. e.,

$$\omega_L\left(Z^b, X\right) = \left[S^*\left(d\psi^b\right)\right](X), \ \forall X \in \mathfrak{X}(TQ).$$

So, the compatibility condition is equivalent to the regularity of $\mathscr{C}$ (independently of the choice of the constraint functions).

Ideal constraints. A constraint function is said to be ideal if the work of reaction forces of the constraint is equal to zero [28, 29]. A constrained Lagrangian system is called ideal if all the constraints are ideal. Notice that a constrained Lagrangian system is ideal if and only if the Liouville vector field $\Delta = \dot{q}^i\partial/\partial\dot{q}^i$ is tangent to the constraint submanifold N. In local charts, this implies that, for every ideal constraint function ψ^a, the term

$$\dot{q}^i \frac{\partial\psi^a(q)}{\partial\dot{q}^i} = 0$$

vanishes identically on the constraint submanifold N. An ideal system is energy preserving, that is, E_L is a conserved quantity for the solution $\Gamma_{L,N}$ of Eq. (2.9), i. e.,

$$\Gamma_{L,N}(E_L) = 0. \tag{2.12}$$

Subsequently, we will always assume that all the constraints are ideal. Notice that, if the constraint functions depend on velocity variables linearly, then they are ideal.

Constrained dynamics as a projection. Consider the distribution F_L on TQ along N whose annihilator is the codistribution F^o_L defined in Eq. (2.8). Note that, at each point v in $N \subseteq TQ$, the subspace $(F_L)_v$ is a coisotropic subspace of the symplectic vector space $\left(T_v TQ, \omega_L(v)\right)$. If the

compatibility condition (2.11) is satisfied, then the distribution $\mathscr{H}$, which is defined to be the intersection

$$\mathscr{H} := TN \cap F_L, \tag{2.13}$$

becomes a symplectic subbundle of the symplectic vector bundle T_NTQ. The converse of this assertion is also true. That is, if $\mathscr{H}$ is a symplectic subbundle, then the constraint system satisfies the compatibility condition [18]. We denote the restriction of ω_L to $\mathscr{H}$ by $\omega_{\mathscr{H}}$. Analogously, we restrict dE_L to $\mathscr{H}$ as a one-form $dE_L|_{\mathscr{H}}$ on $\mathscr{H}$. In this case, the unique solution Γ_H of the symplectic equation

$$\iota_{\Gamma_H}\omega_{\mathscr{H}} = dE_L|_{\mathscr{H}}$$

is the vector field $\Gamma_{L,N}$ generating the constrained dynamics. To prove this fact, we only have to restrict Eq. (2.9) to $\mathscr{H}$ and use the nondegeneracy property of $\omega_{\mathscr{H}}$ (for the linear case, see [4, 30]). Further, we have the following direct sum decomposition:

$$TTQ|_N = \mathscr{H} \oplus \mathscr{H}^{\perp}. \tag{2.14}$$

This decomposition enables us to define two new complementary projectors

$$\bar{\mathscr{P}} : TTQ|_N \mapsto \mathscr{H}, \qquad \bar{\mathscr{Q}} : TTQ|_N \mapsto \mathscr{H}^{\perp}.$$

Note that we have the relation $\bar{\mathscr{P}}(\Gamma_L) = \Gamma_{L,N}$ between the unconstrained Euler–Lagrange vector field Γ_L and the constrained one $\Gamma_{L,N}$.

The case of linear constraints. A particularly important example of nonholonomic Lagrangian system is when the constraint manifold N is a linear subbundle of the tangent bundle TQ. In this case, the constraint functions $\{\psi^a\}$ turn out to be

$$\psi^a(q,\dot{q}) = \psi^a_i(q)\dot{q}^i. \tag{2.15}$$

So that we have the following linear constraint submanifold:

$$N_\ell = \left\{(q^i,\dot{q}^j) \in TQ : \psi^a_i(q)\dot{q}^i = 0\right\}. \tag{2.16}$$

Throughout the text, we denote a linear constraint submanifold by N_ℓ. Thus, the notation N_ℓ is reserved for the linear constraint submanifold presented in (2.16). *Fiber-wise* linearity of the constraint functions ψ^a enables us to define the following one-forms:

$$\overline{\psi}^a(q) = \psi^a_i(q)dq^i \in \Lambda^1(Q). \tag{2.17}$$

It is easy to realize that the codistribution N^o_ℓ, defined to be the annihilator of N_ℓ, is (locally) generated by the one-forms $\overline{\psi}^a$. Then, by pulling the one-forms $\{\overline{\psi}^a\}$ back to TQ with the help of the tangent bundle projection τ_Q, we arrive at one-forms

$$\tau^*_Q\overline{\psi}^a(q) = \psi^a_i(q)dq^i = S^*\left(d\psi^a\right) \in \Lambda^1(TQ). \tag{2.18}$$

Hence,

$$\tau^*_Q\left(N^o_\ell\right) = S^*\left(TN^o_\ell\right).$$

This immediately implies that, in this case, the admissibility condition is automatically satisfied. Note that, even though the one-form sections presented in Eq. (2.17) and Eq. (2.18) locally look identical, they are not. The former, $\{\overline{\psi}^a\}$, are one-form sections on Q taking values in T^*Q, whereas the latter, $\{\tau^*_Q\overline{\psi}^a\}$, are defined on TQ and take values in T^*TQ.

Legendre transformation of constraint Lagrangian dynamics. An unconstrained Hamiltonian system is defined by a triple (T^*Q,ω_Q,H). Denote the Hamiltonian vector field of H by Γ_H, i. e., Γ_H is the unique solution of Hamilton's equation

$$\iota_{\Gamma_H}\omega_Q = dH. \tag{2.19}$$

We define a nonholonomic Hamiltonian system by the Legendre transformation of a nonholonomic Lagrangian system. Consider a Lagrangian function L and a constraint submanifold N in the form (2.6). Define a submanifold M of T^*Q as the image space $\mathbb{F}L(N)$. The equation of motion associated to the constrained Hamiltonian system is defined to be

$$\left(\iota_{\Gamma_{H,M}}\omega_Q - dH\right)\big|_M \in F^o, \qquad \Gamma_{H,M}|_M \in TM, \tag{2.20}$$

where F^o is a codistribution on T^*Q obtained by the pushing F_L forward by $\mathbb{F}(L)$.

2.2. Hamilton – Jacobi Theory for Systems with Linear Constraints

We start with a regular Lagrangian L on TQ and nonholonomic linear constraints defined by the vector subbundle N_ℓ presented in (2.16). The constraint functions $\psi_i^a(q)$ lead to the determination of the set of differential one-forms $\overline{\psi^a}$ on Q as given in Eq. (2.17). Note that the image space of these one-forms determines a subbundle N_ℓ^0 of the cotangent bundle T^*Q annihilating N_ℓ. Define an ideal

$$\mathscr{I}(N_\ell^0) = \left\{\beta_a \wedge \overline{\psi^a} : \beta_a \in \Lambda(Q)\right\} \tag{2.21}$$

of the exterior algebra $\Lambda(Q)$. A vector field X on Q is called a characteristic vector field of the ideal if it satisfies $\iota_X(\overline{\psi^a}) = 0$ for all $\overline{\psi^a}$. A characteristic vector field X of $\mathscr{I}(N_\ell^0)$ preserves the ideal, that is,

$$\iota_X \mathscr{I}(N_\ell^0) \subset \mathscr{I}(N_\ell^0). \tag{2.22}$$

Notice that a vector field X taking values in the constraint subbundle N_ℓ is a characteristic vector field of the ideal.

We are ready now to state the Hamilton – Jacobi theorem for linear nonholonomic Lagrangian systems [10], see also [21, 31]. Let X be a vector field on Q such that $X(Q) \subset N_\ell$, and let the constrained Euler – Lagrange vector field $\Gamma_{L,N}$ be the one given in Eq. (2.9). We define a vector field $\Gamma_{L,N_\ell}^X \in \mathfrak{X}(Q)$ as follows:

$$\Gamma_{L,N_\ell}^X := T\tau_Q \circ \Gamma_{L,N_\ell} \circ X. \tag{2.23}$$

So the diagram

$$\begin{array}{ccc} N_\ell \subset TQ & \xrightarrow{\Gamma_{L,N_\ell}} & TTQ \\ {\scriptstyle X}\uparrow \;\downarrow{\scriptstyle \tau_Q} & & \downarrow{\scriptstyle T\tau_Q} \\ Q & \xrightarrow{\Gamma_{L,N_\ell}^X} & TQ \end{array} \tag{2.24}$$

is commutative.

Theorem 1. *Let X be a vector field on Q such that $X(Q) \subset N_\ell$ and $d(\mathbb{F}L \circ X) \in \mathscr{I}(N_\ell^0)$. Then the following conditions are equivalent:*

(*i*) Γ_{L,N_ℓ}^X *and* Γ_{L,N_ℓ} *are X-related;*

(*ii*) $d(E_L \circ X) \in N_\ell^0$.

Completely nonholonomic systems. Let us now introduce a special case of nonholonomic linear constraints which turns the condition $d(E_L \circ X) \in N_\ell^0$ into the identity

$$d\left(E_L \circ X\right) = 0.$$

Definition 1. A distribution $\mathscr{D} \leqslant TQ$ is said to be completely nonholonomic (or bracket-generating) if $\mathscr{D}$ along with all of its iterated Lie brackets $[\mathscr{D}, \mathscr{D}], [\mathscr{D}, [\mathscr{D}, \mathscr{D}]], \ldots$ spans TQ.

An important result related to this kind of distribution is the following.

Theorem 2 (Chow – Rashevskii theorem). *Let Q be a connected differentiable manifold. If a distribution $\mathscr{D} \leqslant TQ$ is completely nonholonomic, then any two points on Q can be joined by a horizontal path γ, i. e., the derivative of γ is tangent to $\mathscr{D}$.*

A detailed proof of this theorem for regular distributions can be found in [32]. For singular distributions, see [33]. A particularly important consequence of this theorem is given by the following result [22].

Proposition 1. *Let Q be a connected differentiable manifold and $\mathscr{D} \leqslant TQ$ be a completely nonholonomic distribution. Then there is no nonzero exact one-form in the annihilator $\mathscr{D}^o \leqslant T^*Q$.*

It is important to remark that this proposition can be analogously proved for the case in which $\mathscr{D}$ is a singular distribution by taking into account the Chow – Rashevskii theorem for singular distributions [33]. So, as an immediate consequence we have the following theorem, which has been proved by T. Ohsawa and A. Bloch in [22].

Theorem 3. *Assume that the distribution defined by the constraint vector bundle N_ℓ is completely nonholonomic. Let X be a vector field on Q such that $X(Q) \subset N_\ell$ and $d(\mathbb{F}L \circ X) \in \mathscr{I}(N_\ell^0)$. Then the following conditions are equivalent:*

(*i*) Γ^X_{L,N_ℓ} *and* Γ_{L,N_ℓ} *are X-related;*

(*ii*) $d(E_L \circ X) = 0$.

Legendre Transformations of Theorems (1) **and** (3)**.** Define $M = \mathbb{F}L(N_\ell)$. For a fixed one-form σ on Q such that $\sigma\,(Q) \subseteq M$ we will define a vector field $\Gamma^\sigma_{H,M}$ on Q by satisfying the following diagram [10].

$$\begin{array}{ccc} M \subset T^*Q & \xrightarrow{\Gamma_{H,M}} & TT^*Q \\ \sigma \uparrow \downarrow \pi_Q & & \downarrow T\pi_Q \\ Q & \xrightarrow{\Gamma^\sigma_{H,M}} & TQ \end{array} \tag{2.25}$$

Theorem 4. *Let σ be a 1-form on Q such that $\sigma(Q) \subset M$ and $d\sigma \in \mathscr{I}(N_\ell^0)$. Then the following conditions are equivalent:*

(*i*) $\Gamma^\sigma_{H,M}$ *and* $\Gamma_{H,M}$ *are σ-related;*

(*ii*) $d(H \circ \sigma) \in N_\ell^0$.

Notice that this theorem only holds for a constraint manifold $M \subseteq T^*Q$ which comes from a linear constraint manifold $N_\ell \subseteq TQ$, via a regular Lagrangian L. Finally, let us present a Hamiltonian version of Theorem 3.

Theorem 5. *Assume that the distribution defined by the constraint vector bundle N_ℓ is completely nonholonomic. Let σ be a 1-form on Q such that $\sigma(Q) \subset M$ and $d\sigma \in \mathscr{I}(N_\ell^0)$. Then the following conditions are equivalent:*

(*i*) $\Gamma^\sigma_{H,M}$ *and* $\Gamma_{H,M}$ *are σ-related;*

(*ii*) $d(H \circ \sigma) = 0$.

2.3. Hamilton – Jacobi Theory for Systems with Nonlinear Constraints

This subsection gathers our first novel contribution in this paper: We generalize the Hamilton – Jacobi theorems (1) and (4) for the case of nonlinear constraints. To depict it, recall the constraint submanifold N in Eq. (2.6) with (possibly nonlinear) constraint functions ψ^a and the codistribution F_L^o defined in Eq. (2.8) generated by the differential one-forms $S^*(d\psi^a) = (\partial\psi^a/\partial\dot{q}^i)dq^i$ on TQ. We introduce the ideal $\mathscr{I}(F_L^o)$ generated by these one-forms

$$\mathscr{I}(F_L^o) = \{\beta_a \wedge S^*(d\psi^a) : \beta_a \in \Lambda^s(TQ)\}. \tag{2.26}$$

Consider the Euler – Lagrange vector field $\Gamma_{L,N}$ for the constraint system. Let X be a vector field on Q such that $X(Q) \subset N$ and define a vector field $\Gamma_{L,N}^X$ by the composition $T\tau_Q \circ \Gamma_{L,N} \circ X$. Notice that this definition coincides with the one presented in diagram (2.24) by replacing the linear constraint submanifold N_ℓ with an arbitrary submanifold N.

Theorem 6. *Let X be a vector field on Q such that $X(Q) \subset N$ and $(X \circ \tau_Q)^* \omega_L \in \mathscr{I}(F_L^o)$. Then the following conditions are equivalent:*

(*i*) $\Gamma_{L,N}^X$ *and* $\Gamma_{L,N}$ *are* X*-related*;

(*ii*) $d\big(E_L \circ (X \circ \tau_Q)\big) \in F_L^o$.

We first present the following two lemmas that we shall need while proving Theorem (6).

Lemma 1. *Consider the symplectic manifold (TQ, ω_L) with a regular Lagrangian L, then*

$$\omega_L(U, W) = 0, \qquad \text{if} \quad U, W \in \ker T\tau_Q. \tag{2.27}$$

Proof. Let α_L be the Poincaré – Cartan one-form on Q presented in (2.3). Since α_L is a semibasic one-form, we have that for each two vertical (local) vector fields W and U on TQ

$$\omega_L(U, W) = -d\alpha_L(U, W) = -U\big(\alpha_L(W)\big) + W\big(\alpha_L(U)\big) + \alpha_L\big([U, W]\big) = 0.$$

Notice that the Lie bracket of vector fields preserves vertical vector fields. □

Lemma 2. *The identities*

$$(X \circ \tau_Q)^* F_L^o = F_L^o, \qquad (X \circ \tau_Q)_* F_L = F_L \tag{2.28}$$

hold along X.

Proof. The mapping $X \circ \tau_Q$ reduces to the identity mapping when it is restricted to the base manifold Q. This gives that, along X, the pullback of F_L^o turns out to be equal to F_L^o since it is composed of semisimple one-forms on TQ. It is possible to observe this in terms of coordinates as follows:

$$(X \circ \tau_Q)^* \left[\left(\frac{\partial\psi^a}{\partial\dot{q}^i}\right) dq^i\right] = \left(\frac{\partial\psi^a}{\partial\dot{q}^i} \circ X \circ \tau_Q\right) d\left(q^i \circ X \circ \tau_Q\right) = \left(\frac{\partial\psi^a}{\partial\dot{q}^i} \circ X \circ \tau_Q\right) dq^i.$$

Take a generic vector field Z taking values in F_L and consider its restriction to the image space of X. According to Lemma (2), if α is in F_L^o, then so does $(X \circ \tau_Q)^* \alpha$. This manifests that its pushforward $(X \circ \tau_Q)_* Z$ is an element of F_L. □

Proof (*of Theorem* 6). First notice that the constrained Euler – Lagrange vector field $\Gamma_{L,N}$ takes values in F_L. So that it is a characteristic vector field of the ideal $\mathscr{I}(F_L^o)$ presented in Eq. (2.26), i. e.,

$$\iota_{\Gamma_{L,N}} \mathscr{I}(F_L^o) \subseteq \mathscr{I}(F_L^o).$$

In the statement of Theorem 6 we have assumed that $(X \circ \tau_Q)^* \omega_L \in \mathscr{I}(F_L^o)$. Therefore, we have that

$$\iota_{\Gamma_{L,N}} (X \circ \tau_Q)^* \omega_L \in F_L^o. \tag{2.29}$$

By taking the pullback of both sides of the equality in (2.9) with the mapping $X \circ \tau_Q$ we find that

$$(X \circ \tau_Q)^* \left[\iota_{\Gamma_{L,N}} \omega_L\right] \in F_L^o \tag{2.30}$$

if and only if

$$(X \circ \tau_Q)^* \, dE_L = d\left(E_L \circ X \circ \tau_Q\right) \in F_L^o. \tag{2.31}$$

This gives that the second condition, (ii), can be equivalently written as Eq. (2.30).

$(i) \implies (ii)$: Let W be an arbitrary vector field on TQ, and compute the following:

$$\begin{aligned}
(X \circ \tau_Q)^* \left(\iota_{\Gamma_{L,N}} \omega_L\right)(W) &= \tau_Q^* X^* \left(\iota_{\Gamma_{L,N}} \omega_L\right)(W) = X^* \left(\iota_{\Gamma_{L,N}} \omega_L\right)(T\tau_Q \cdot W) \\
&= \iota_{\Gamma_{L,N}^X} X^* \omega_L \left(T\tau_Q \cdot W\right) = X^* \omega_L \left(\Gamma_{L,N}^X, T\tau_Q \cdot W\right) \\
&= X^* \omega_L \left(T\tau_Q \circ \Gamma_{L,N} \circ X, T\tau_Q \cdot W\right) = \tau_Q^* X^* \omega_L \left(\Gamma_{L,N} \circ X, W\right) \\
&= (X \circ \tau_Q)^* \, \omega_L \left(\Gamma_{L,N} \circ X, W\right),
\end{aligned}$$

where we have used the X-relatedness of $\Gamma_{L,N}^X$ and $\Gamma_{L,N}$ in the third line, whereas we have used the definition of $\Gamma_{L,N}^X$ in the fifth line. Restrict this calculation on the image space of X so that

$$(X \circ \tau_Q)^* \left(\iota_{\Gamma_{L,N}} \omega_L\right)(W) = (X \circ \tau_Q)^* \, \omega_L \left(\Gamma_{L,N} \circ X, W\right) = \iota_{\Gamma_{L,N}} \left(X \circ \tau_Q\right)^* \omega_L(W),$$

for an arbitrary vector field W. Therefore,

$$(X \circ \tau_Q)^* \left(\iota_{\Gamma_{L,N}} \omega_L\right) = \iota_{\Gamma_{L,N}} \left(X \circ \tau_Q\right)^* \omega_L$$

along X. Since the term on the right-hand side of this identity is an element of F_L^o, we see that

$$(X \circ \tau_Q)^* \left(\iota_{\Gamma_{L,N}} \omega_L\right) \in F_L^o.$$

This is an equivalent representation of the condition (ii) given in Eq. (2.30). So we have proved that (i) implies (ii).

$(ii) \implies (i)$: Consider the constrained vector field $\Gamma_{L,N}$ and an arbitrary vector field V taking values in F_L. Instead of the whole TQ, we are only interested in the image space of X. Introduce the following vector fields along X:

$$\Lambda := \Gamma_{L,N} - T\left(X \circ \tau_Q\right) \cdot \Gamma_{L,N}, \qquad \Sigma := V - T\left(X \circ \tau_Q\right) \cdot V. \tag{2.32}$$

Notice that both of the vectors $\Lambda\big(X(q)\big)$ and $\Sigma\big(X(q)\big)$ are in the kernel of $T\tau_Q$, so that, according to Lemma (1), at $X(q)$, we compute

$$\omega_L(\Lambda, \Sigma)|_{X(q)} = 0. \tag{2.33}$$

Substitute now the definitions of the vector fields in Eq. (2.32) into identity (2.33). This implies that

$$\begin{aligned}
\omega_L(\Lambda, \Sigma)|_{X(q)} &= \omega_L(\Gamma_{L,N}, V)|_{X(q)} - \omega_L\big(\Gamma_{L,N}, T(X \circ \tau_Q) \cdot V\big)|_{X(q)} \\
&= -\, \omega_L\big(T(X \circ \tau_Q) \cdot \Gamma_{L,N}, V\big)|_{X(q)} \\
&= +\, \omega_L\big(T(X \circ \tau_Q) \cdot \Gamma_{L,N}, T(X \circ \tau_Q) \cdot V\big)|_{X(q)}.
\end{aligned} \tag{2.34}$$

Recall that we assume that $(X \circ \tau_Q)^* \, \omega_L \in \mathscr{I}\left(F_L^o\right)$, which implies Eq. (2.29). This makes the term in the third line of the calculation (2.34) zero. The condition (ii) given in the form of Eq. (2.30) makes the second term on the right-hand side of the first line of the calculation (2.34) zero. Eventually, we are left with the following identity:

$$\omega_L\big(\Gamma_{L,N} - T\left(X \circ \tau_Q\right) \cdot \Gamma_{L,N}, V\big)|_{X(q)} = 0 \tag{2.35}$$

for an arbitrary V in F_L. Hence, we have proved that

$$\big(\Gamma_{L,N} - T(X \circ \tau_Q) \cdot \Gamma_{L,N}\big)|_{X(q)} \in F_L^{\perp}. \tag{2.36}$$

From the compatibility condition (2.11), we have that $\Gamma_{L,N} - T(X \circ \tau_Q) \cdot \Gamma_{L,N}$ is identically zero along X. This is nothing but condition (i).

Definition 2. A vector field X on Q satisfying the conditions of Theorem 6 will be called a solution of the constrained Hamilton – Jacobi problem determined by the constraint submanifold $N \subseteq TQ$ and a regular Lagrangian L.

Notice that, as an immediate consequence of condition (ii) of Theorem 6, we have that any solution X of Hamilton – Jacobi problem without constraints, which satisfies that $X(Q) \subset N$, is a solution of the constrained Hamilton – Jacobi problem determined by the constraint manifold $N \subseteq TQ$ and a regular Lagrangian L.

Corollary 1. *Let X be a vector field on Q in the conditions of Theorem 6 such that $TX(TQ) \subseteq F_L$. Then the following conditions are equivalent:*

(i) $\Gamma_{L,N}^X$ *and* $\Gamma_{L,N}$ *are X-related;*

(ii) $d(E_L \circ X) = 0$.

Proof. Notice that, by using that τ_Q is a submersion, the one-form $d\big(E_L \circ X\big)$ is zero if and only if

$$d\big(E_L \circ (X \circ \tau_Q)\big) = 0.$$

Finally, it is easy to check that the only way in which the one-form $d\big(E_L \circ (X \circ \tau_Q)\big)$ is in the annihilator F_L^o is that it is cancelled. This is an immediate consequence of the condition $TX(TQ) \subseteq F_L$. Consider a vector field X on Q under the conditions of Corollary 1. Then, by taking into account $(X \circ \tau_Q)^* \omega_L \in \mathscr{I}(F_L^o)$, the inclusion $TX(N) \subseteq F_L$ implies that $X^*\omega_L = 0$. In fact, for each a,

$$S^*(d\psi^a)\left(TX\left(\frac{\partial}{\partial q^i}\right)\right) = S^*(d\psi^a)\left(\frac{\partial}{\partial q^i} + \frac{\partial X^k}{\partial q^i}\frac{\partial}{\partial \dot{q}^k}\right) = d\psi^a\left(\frac{\partial}{\partial \dot{q}^i}\right) = \frac{\partial \psi^a}{\partial \dot{q}^i}.$$

Hence, $TX(TQ) \subseteq F_L$ is equivalent to

$$F_L^0 = 0 \;\Leftrightarrow\; F_L = T_N(TQ).$$

Thus, this condition implies that the constraint functions do not produce reaction forces. For this, it is enough to find some vector field X satifying that $TX(TQ) \subseteq F_L$. Indeed, the equation of motion associated to the constraint system is defined to be

$$\iota_{\Gamma_{L,N}}\omega_L - dE_L = 0, \qquad \Gamma_{L,N}|_N \in TN. \tag{2.37}$$

From nonlinear constraints to linear constraints. It is interesting to see how the Hamilton – Jacobi Theorem 6 for nonholonomic systems with nonlinear constraints covers the Hamilton – Jacobi Theorem 1 for nonholonomic systems with linear constraints. Note that, for the linear case,

$$F_L^o = \tau_Q^*(N_\ell^o).$$

This gives that the assumption $(X \circ \tau_Q)^* \omega_L \in \mathscr{I}(F_L^o)$ of Theorem 6 reduces to $X^*\omega_L \in \mathscr{I}(N_\ell^o)$. A direct computation shows us that this is equivalent to the assumption $d(\mathbb{F}L \circ X) \in \mathscr{I}(N_\ell^o)$ of Theorem 1. Indeed, by taking into account that $S^*(dL) = \mathbb{F}L^*\theta_Q$, we compute

$$X^*\omega_L = -X^*\big(dS^*(dL)\big) = -X^*(d\mathbb{F}L^*\theta_Q) = -d(\mathbb{F}L \circ X)^*\theta_Q = -d(\mathbb{F}L \circ X).$$

Analogously, the second condition $d\big(E_L \circ (X \circ \tau_Q)\big) \in F_L^o$ in Theorem 6 reduces to the second condition $d(E_L \circ X) \in N_\ell^o$ in Theorem 1. Thus, Theorem 1 can be seen as a corollary of Theorem 6. Analogously, Theorem 3 is a particular case of Theorem 7.

Completely nonholonomic systems. Let us now give another consequence of Theorem 6 which can give us more interesting examples. In order to do that, we need the Chow – Rashevskii Theorem 2 for the nonlinear case. Notice that, as a first difference with the linear case, F_L is just a distribution

on TQ along N (it is not defined on the whole TQ). Thus, we will solve this problem by defining the following family $\mathscr{S}$ of local one-forms on TQ:

$$\mathscr{S} := \{S^*(\sigma) \in \Lambda^1_{local}(TQ) \ : \ \sigma_{|TN} \equiv 0\}. \tag{2.38}$$

So, we define the codistribution G^o_L on TQ generated by $\mathscr{S}$, i. e., for each $v_q \in T_qQ$ the fiber $G^o_{L|v_q}$ is the vector subspace of $T^*_{v_q}(TQ)$ generated by all the evaluations of the (local) one-forms in $\mathscr{S}$ at v_q. So, G_L will be the distribution on TQ such that its annihilator is G^o_L. Then G_L is a smooth (possibly singular) distribution on TQ. Notice that G_L can be alternatively defined as the smooth distribution generated by the (local) vector fields on TQ such that its restrictions to N are tangent to F_L. Observe that far from N the fibers can be the whole tangent space of TQ. Another remark is that there exists, at least, one global one-form contained in $\mathscr{S}$: the zero section of π_Q. Finally, for all $v_q \in N$,

$$G_{L|v_q} = F_{L|v_q}. \tag{2.39}$$

In fact, each covector $\sigma_{v_q} \in F^o_L$ can be written as a combination

$$\sigma_{v_q} = \lambda_a \left[S^*(d\psi^a)\right](v_q),$$

and all the local one-forms $S^*(d\psi^a)$ are contained in $\mathscr{S}$. Let us consider the family of (local) vector fields $\mathscr{S}^\sharp_N$ tangent to G_L whose domains have nonempty intersections with N. Then F_L is the distribution generated by $\mathscr{S}^\sharp_N$ along N. Thus, F_L is called completely nonholonomic if the iterated Lie bracket of vector fields in $\mathscr{S}^\sharp_N$ generate $T_N(TQ)$. In this case we will say that

$$[F_L, F_L], [F_L, [F_L, F_L]], \ldots$$

spans the vector bundle $T_N(TQ)$. Obviously, G_L is completely nonholonomic if and only if F_L is completely nonholonomic. We are now ready to give a generalization of Theorem 3.

Theorem 7. *Let X be a vector field on Q in the conditions of Theorem 6 such that F_L is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $\Gamma^X_{L,N}$ *and* $\Gamma_{L,N}$ *are X-related;*

(*ii*) $d(E_L \circ X) = 0$.

Proof. Due to the fact that G_L is completely nonholonomic we can use Proposition 1 to have that $d(E_L \circ X) \in G_L$ if and only if $d(E_L \circ X) = 0$. Notice that, obviously, $d(E_L \circ X \circ \tau_Q) \in F^o_L$ if and only if $d(E_L \circ X \circ \tau_Q) \in G^o_L$. □

In an obvious way, we can present its Hamiltonian counterpart by following the notation introduced for Theorem 4.

Theorem 8. *Let σ be a one-form on Q such that $\sigma(Q) \subset M$ and $(\sigma \circ \tau_Q)^* \omega_Q \in \mathscr{I}(F^o)$. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}^\sigma_{H,M}$ *and* $\overline{\Gamma}_{H,M}$ *are σ-related;*

(*ii*) $d(H \circ (\sigma \circ \tau_Q)) \in F^o$.

Here F^o is just the pullback of F^o_L by the inverse of the Legendre transformation $\mathbb{F}L^{-1}$.

Definition 3. A one-form σ on Q satisfying the conditions of Theorem 8 will be called a solution of the constrained Hamilton – Jacobi problem given by the constraint manifold $M \subseteq T^*Q$ and the Hamiltonian H.

So, Corollary 1 has an obvious counterpart on the Hamiltonian side.

Corollary 2. *Let σ be a one-form on Q in the conditions of Theorem 8 such that $T\sigma(TQ) \subseteq F$. Then the following conditions are equivalent:*

(*i*) $\Gamma^{\sigma}_{H,M}$ *and* $\Gamma_{H,M}$ *are* σ*-related;*

(*ii*) $d(H \circ \sigma) = 0$.

The condition $T\sigma(TQ) \subseteq F$ is equivalent to $F = T_M(T^*Q)$ and, hence, the equation of motion associated to the constraint Hamiltonian system is defined to be

$$\iota_{\Gamma_{H,M}}\omega_Q - dH = 0, \qquad \Gamma_{H,M}|_M \in TM. \tag{2.40}$$

Under the Legendre transformation, Theorem 7 turns out to be the following form.

Theorem 9. *Let σ be a one-form on Q in the conditions of Theorem 8 such that G_L is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $\Gamma^{\sigma}_{H,M}$ *and* $\Gamma_{H,M}$ *are* σ*-related;*

(*ii*) $d(H \circ \sigma) = 0$.

3. HAMILTON – JACOBI THEORY FOR REDUCED NONHOLONOMIC MECHANICS

3.1. Reduction of Nonholonomic Systems

Let us assume now that a Lie group G acts freely and properly on the base manifold Q leaving invariant the Lagrangian L and the constraint submanifold N presented in Subsection (2.1). So, the following result can be proved [28]: Under the above hypotheses, the geometric objects defined in (2.1), namely, α_L, ω_L, E_L, $\mathbb{F}L$, Γ_L and F_L, are G-invariant. These imply that the vector field $\Gamma_{L,N}$, satisfying (2.9), is G-invariant.

The quotient of an object A by G will be denoted by the name of the overlined quantity, i. e., $\overline{A}$. So, let us define the following projections and reduced spaces:

$$\rho_Q : Q \to \overline{Q}, \qquad \rho_{TQ} : TQ \to \overline{TQ}, \qquad \rho_N : N \to \overline{N} \tag{3.1}$$

which are the principal bundles associated with the actions of the group G with Q, TQ and N, respectively. Consider the tangent lift of the projection ρ_{TQ}, that is, $T\rho_{TQ}$ a linear mapping from TTQ to $T\overline{TQ}$ and its kernel is a subbundle of TTQ consisting of the vertical vectors. We denote the vectical bundle by $\mathscr{V}$. Similarly, we denote the kernel of the tangent lift $T\rho_N$ of the projection ρ_N by $\mathscr{V}_N$. Notice that $\mathscr{V}_N$ is, in fact, the restriction of $\mathscr{V}$ to TN. Let us study the projection $\overline{F}^o_L$ of F^o_L. Notice that, because of the invariance, we can define the projection $\overline{S}$ of S through the following diagram.

$$\begin{array}{ccc} T(TQ) & \xrightarrow{\quad S \quad} & T(TQ) \\ \Big\downarrow{\scriptstyle T\rho_{TQ}} & & \Big\downarrow{\scriptstyle T\rho_{TQ}} \\ T\overline{TQ} & \xrightarrow{\quad \overline{S} \quad} & T\overline{TQ} \end{array} \tag{3.2}$$

Then it is an easy computation to show that

$$\overline{F}^o_L = \overline{S}^*\left(T\overline{N}^o\right). \tag{3.3}$$

Notice that $\overline{F}^o_L$ is characterized by the following identity:

$$\rho^*_{TQ}\overline{F}^o_L = F^o_L. \tag{3.4}$$

It is important to remark that, projecting directly Eq. (2.9), we obtain that

$$\iota_{\overline{\Gamma}_{L,N}}\overline{\omega}_L - d\overline{E_L} \in \overline{F}_L^o, \quad \overline{\Gamma}_{L,N}|_{\overline{N}} \in T\overline{N}. \tag{3.5}$$

Recall the symplectic distribution $\mathscr{H}$ on TQ defined in Eq. (2.13). We define a distribution on TQ along N by

$$\mathscr{U}_v = \{Y \in \mathscr{H}_v : \omega_L(Y,Z)(v) = 0,\ \forall Z \in \mathscr{V}_v \cap (F_L)_v\}. \tag{3.6}$$

In other words, $\mathscr{U} = \mathscr{H} \cap (\mathscr{V} \cap F_L)^{\perp}$. One can easily see that the symplectic form ω_L and the one-form dE_L can be restricted to $\mathscr{U}$ as a two-form $\omega_{\mathscr{U}}$ on $\mathscr{U}$ and a one-form $dE_{L\mathscr{U}}$, respectively. $\mathscr{U}$ projects to the distribution $\overline{\mathscr{U}}$ on $\overline{TQ}$ along $\overline{N}$. Notice also that the objects $\omega_{\mathscr{U}}$ and $dE_{L\mathscr{U}}$ are also reducible. We denote the reduced forms by $\omega_{\overline{\mathscr{U}}}$ and $d\overline{E_L}_{\overline{\mathscr{U}}}$, respectively. The following theorem states the reduced dynamics.

Theorem 10. *Let $\Gamma_{L,N}$ be the solution of the nonholonomic Lagrangian system. Then $\overline{\Gamma}_{L,N}$ is a section of $\overline{\mathscr{U}}$ satisfying*

$$\iota_{\overline{\Gamma}_{L,N}}\omega_{\overline{\mathscr{U}}} = d\overline{E_L}_{\overline{\mathscr{U}}}. \tag{3.7}$$

It is important to notice that the pair $(\overline{\mathscr{U}}, \omega_{\overline{\mathscr{U}}})$ is a symplectic generalized vector bundle on $\overline{N}$, i. e., for each $\overline{v} \in \overline{\mathscr{U}}$ we have that $\omega_{\overline{\mathscr{U}}\,\overline{v}}$ is a nondegenerate two-form on $\overline{\mathscr{U}}_{\overline{v}}$. Furthermore, it satisfies the following equality:

$$d\overline{E_L}_{\overline{\mathscr{U}}} = \left(d\overline{E_L}_{|\overline{N}}\right)_{\overline{\mathscr{U}}}. \tag{3.8}$$

This reduction was first introduced by L. Bates and J. Sniatycki in [4] for the linear case. However, it is straighforward to prove that Theorem 10 is also valid for the nonlinear case.

3.2. Hamilton – Jacobi Theory

Invariant vector fields on Q and sections of a reduced bundle. Let $\overline{X}$ be a section of the projection $\overline{\tau}_Q : \overline{TQ} \to \overline{Q}$ such that $\overline{X}\left(\overline{Q}\right) \subseteq \overline{N}$. Then we may define a vector field $\overline{\Gamma}_{L,N}^{\overline{X}}$ on $\overline{Q}$ by

$$\overline{\Gamma}_{L,N}^{\overline{X}} = T\overline{\tau}_Q \circ \overline{\Gamma}_{L,N} \circ \overline{X}. \tag{3.9}$$

Hence, the following diagram is commutative.

$$\begin{array}{ccc} \overline{N} & \xrightarrow{\overline{\Gamma}_{L,N}} & T\left(\overline{TQ}\right) \\ {\scriptstyle \overline{X}}\uparrow & & \downarrow{\scriptstyle T\overline{\tau}_Q} \\ \overline{Q} & \xrightarrow{\overline{\Gamma}_{L,N}^{\overline{X}}} & T\overline{Q} \end{array} \tag{3.10}$$

Let us study carefully the section $\overline{X}$. Notice that, taking into account that the action on Q is free, to each $q \in Q$ there corresponds only one tangent vector X_q in T_qQ such that

$$\overline{X}\left(\overline{q}\right) = \rho_{TQ}\left(X_q\right). \tag{3.11}$$

This observation enables us to define a map $X : Q \to TQ$ according to the identity $X\left(q\right) = X_q$ for all $q \in Q$. By uniqueness, the map X is G-invariant. Now, to prove that X is a *bona-fide* vector

field, we need to show that it is differentiable. To this end, consider two (local) sections Θ and θ of ρ_{TQ} and ρ_Q, respectively, such that the following diagram is commutative.

$$\begin{array}{ccc} \overline{TQ} & \xrightarrow{\quad\Theta\quad} & TQ \\ \Big\downarrow{\scriptstyle \overline{\tau}_Q} & & \Big\downarrow{\scriptstyle \tau_Q} \\ \overline{Q} & \xrightarrow{\quad\theta\quad} & Q \end{array} \tag{3.12}$$

On the other hand, we know that

$$\rho_{TQ} \circ X = \overline{X} \circ \rho_Q. \tag{3.13}$$

So, the map X^{Θ} defined by the composition $\Theta \circ \rho_{TQ} \circ X$ is differentiable. However, X^{Θ} is not a vector field and, obviously, $X^{\Theta} \neq X$. In fact, X^{Θ} is constant on the orbits of the action of G over Q. By using that θ is a section, $\theta\left(\overline{Q}\right)$ is a submanifold of Q. Hence, the restriction X^{θ} of X^{Θ} to $\theta\left(\overline{Q}\right)$ is differentiable. In fact, by uniqueness $X^{\theta} = X_{|\theta(\overline{Q})}$. So, using the multiplication of the action, one finds that X is differentiable and, hence, X is a G-invariant vector field on Q such that it satisfies Eq. (3.13). Equivalently, we have proved that there exists a one-to-one correspondence

$$\mathfrak{X}_G(Q) \longrightarrow \Gamma\left(\overline{\tau}_Q\right) : X \mapsto \overline{X} \tag{3.14}$$

where $\mathfrak{X}_G(Q)$ is the space of G-invariant vector fields on Q and $\Gamma\left(\overline{\tau}_Q\right)$ is the space of sections of ρ_{TQ}. We will use this correspondence throughout the rest of the paper.

Hamilton – Jacobi equation for reduced nonholonomic systems. Now we are ready to state the following theorem, which is a reduced version of Theorem 6. To do this, we will define the ideal $\mathscr{I}(\overline{F}_L^o)$ in the same way as $\mathscr{I}(F_L^o)$ (see Eq. (2.26)).

Theorem 11. *Let $\overline{X}$ be a section of $\overline{\tau}_Q$ such that $\overline{X}\left(\overline{Q}\right) \subset \overline{N}$ and $\left(\overline{X} \circ \overline{\tau}_Q\right)^* \overline{\omega}_L \in \mathscr{I}\left(\overline{F}_L^o\right)$. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(*ii*) $d\left(\overline{E}_L \circ \left(\overline{X} \circ \overline{\tau}_Q\right)\right) \in \overline{F}_L^o$.

Proof. We will use the correspondence between G-invariant vector fields on Q and sections of $\overline{\tau}_Q$, $X \mapsto \overline{X}$, and Theorem 6 to prove this. So, let X be the G-invariant vector field on Q which projects on $\overline{X}$. Then

$$\overline{X}\left(\overline{Q}\right) = \rho_{TQ}\big(X(Q)\big).$$

So, condition $\overline{X}\left(\overline{Q}\right) \subset \overline{N}$ turns into the following condition:

$$\rho_{TQ}\big(X(Q)\big) \subset \rho_{TQ}(N).$$

By the G-invariance of N we have that this is equivalent to $X(Q) \subset N$. Next, let us work with the condition $\left(\overline{X} \circ \overline{\tau}_Q\right)^* \overline{\omega}_L \in \mathscr{I}\left(\overline{F}_L^o\right)$, i. e.,

$$\left(\overline{X} \circ \overline{\tau}_Q\right)^* \overline{\omega}_L = \overline{\beta}_a \wedge \overline{S}^*\left(\overline{d\psi^a}\right)$$

with $\overline{\beta}_a \in \Lambda^1\left(\overline{TQ}\right)$. Then, taking into account that ρ_{TQ} is a submersion, this easily implies that $(X \circ \tau_Q)^* \omega_L \in \mathscr{I}\left(F_L^o\right)$. So, we are in the conditions of Theorem 6. It is possible to show directly that conditions (*i*) and (*ii*) in the present theorem are equivalent to conditions (*i*) and (*ii*) of Theorem 6. □

Definition 4. A section $\overline{X}$ of $\overline{\tau}_Q$ satisfying the conditions of Theorem 11 will be called a solution of the reduced constrained Hamilton – Jacobi problem given by the constraint manifold $N \subseteq TQ$, the regular Lagrangian L and the action of the group G on Q.

According to the proof of Theorem (11), we actually have a result giving the relation between G-invariant solutions of the Hamilton – Jacobi problem and solutions of the reduced constrained Hamilton – Jacobi problem.

Proposition 2. *Let X be a G-invariant vector field on Q. Then X is a solution of the constrained Hamilton – Jacobi problem if and only if $\overline{X}$ is a solution of the reduced constrained Hamilton – Jacobi problem.*

Notice that at the beginning of this subsection we have given a constructive way to obtain the associated G-invariant vector field X on Q to a section $\overline{X}$ of $\overline{\tau}_Q$. In fact, to have the sections θ and Θ in the diagram (3.12), we only need to use a principal connection on ρ_Q. So, this process defines a detailed way of reconstruction from solutions of the reduced constrained Hamilton – Jacobi problem to the solutions of the constrained Hamilton – Jacobi problem.

Corollary 3. *Let $\overline{X}$ be a section of $\overline{\tau}_Q$ in the conditions of Theorem* 11 *satisfying $T\overline{X}\left(T\overline{Q}\right) \subseteq \overline{F}_L$. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(*ii*) $d\left(\overline{E}_L \circ \overline{X}\right) = 0$.

This result is just a reduced version of Corollary 1. The condition $T\overline{X}\left(T\overline{Q}\right) \subseteq \overline{F}_L$ is actually

$$T\rho_{TQ}\big(TX\left(TQ\right)\big) \subseteq \overline{F}_L.$$

However, the associated G-invariant vector field X does not have to satisfy the conditions of Corollary 1. So, it is curious how in the reduced version we obtain new conditions which are not so restrictive (remember that the conditions of Corollary 1 are only possible in a constrained system in which the constraints do not produce reaction forces).

Hamilton – Jacobi equation for reduced completely nonholonomic systems. By recalling the notation used in Theorem 7, notice that the distribution G_L, satisfying (2.39), is G-invariant. In fact, we only have to prove that the set of (local) one-forms $\mathscr{S}$, in (2.38), is invariant by the action. Thus, we may define the quotient distribution $\overline{G}_L$ on $\overline{TQ}$. Obviously $\overline{G}_L$ restricts to $\overline{F}_L$ along $\overline{N}$. Observe that $\overline{G}_L$ could be alternatively defined following the same way as G_L by using Eq. (3.3). Again we will define the completely nonholonomic condition on $\overline{F}_L$. So, consider the family of (local) vector fields $\overline{\mathscr{S}}^{\sharp}_{\overline{N}}$ on $\overline{TQ}$ tangents to $\overline{G}_L$ whose domains have nonempty intersection with $\overline{N}$. Then $\overline{F}_L$ can be seen as the distribution generated by $\overline{\mathscr{S}}^{\sharp}_{\overline{N}}$ along $\overline{N}$. Thus, $\overline{F}_L$ is called completely nonholonomic if the reiterated Lie bracket of vector fields in $\overline{\mathscr{S}}^{\sharp}_{\overline{N}}$ generates $T_{\overline{N}}\left(\overline{TQ}\right)$. In this case we will say that

$$\left[\overline{F}_L, \overline{F}_L\right], \left[\overline{F}_L, \left[\overline{F}_L, \overline{F}_L\right]\right], \ldots$$

spans the vector bundle $T_{\overline{N}}\left(\overline{TQ}\right)$. Obviously, $\overline{G}_L$ is completely nonholonomic if and only if $\overline{F}_L$ is completely nonholonomic. So, in an analogous way to Theorem 7 we have the following result.

Theorem 12. *Let $\overline{X}$ be a section of $\overline{\tau}_Q$ in the conditions of Theorem* 11 *such that $\overline{F}_L$ is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(*ii*) $d\left(\overline{E}_L \circ \overline{X}\right) = 0$.

The following proposition establishes the relation between the distribution F_L and its projection $\overline{F}_L$.

Proposition 3. *F_L is completely nonholonomic if and only if $\overline{F}_L$ is completely nonholonomic.*

Proof. Remember that F_L (resp. $\overline{F}_L$) is generated by the family of local vector fields $\mathscr{S}_N^\sharp$ (resp. $\overline{\mathscr{S}^\sharp}_{\overline{N}}$) restricted to N (resp. $\overline{N}$). Furthermore, by taking into account (see Eq. 3.4) that

$$\overline{F}_L = T\rho_{TQ}\left(F_L\right)$$

we can find a family of G-invariant vector fields which generates $\mathscr{S}_N^\sharp$. Now suppose first that $\overline{F}_L$ is completely nonholonomic. Let V_{v_q} be a vector in $T_{v_q}\left(TQ\right)$. Fix a (local) G-invariant vector field X on TQ such that

$$X\left(v_q\right) = V_{v_q}.$$

Then the projection $\overline{X}$ of X can be obtained by a linear combination of Lie brackets of (local) vector fields $\overline{X}^a$ on $\overline{TQ}$ tangent to $\overline{F}_L$. Hence, using the associated vector fields X^a of $\overline{X}^a$, we write $X\left(v_q\right)$ as a linear combination of Lie brackets of (local) vector fields tangent to F_L. So, F_L is completely nonintegrable. Notice that the projection of G-invariant vector fields on TQ into sections of $\overline{\tau}_Q$ preserves the Lie bracket. Conversely, assume that F_L is completely nonholonomic. Suppose that we have three (local) G-invariant vector fields $X, Y, Z \in \mathfrak{X}\left(TQ\right) - \{0\}$ and a function $f \in \mathscr{C}^\infty\left(TQ\right)$ such that

$$[X, fY] = Z.$$

Then, for all v_q in the domain of the vector fields and for all $g \in G$,

$$X\left(v_q\right)\left(f \circ \phi_q^T\right) = X\left(v_q\right)\left(f\right),$$

where ϕ is the action of the group G and ϕ^T is its tangent lift. Using this fact and taking into account that $\mathscr{S}_N^\sharp$ is generated by (local) G-invariant vector fields on TQ, we can prove that any tangent vector in $T_N\left(TQ\right)$ can be written as a linear combination of Lie brackets of (local) G-invariant vector fields. Then, projecting these vector fields, we have the result. □

Legendre transformation of the Hamilton – Jacobi equation for reduced nonholonomic systems. Using the Legendre transformation $\mathbb{F}L$, we define a submanifold $M = \mathbb{F}L\left(N\right)$, and a Hamiltonian function $H = E_L \circ \mathbb{F}L^{-1}$ on T^*Q. Let G act on Q and assume that L and N are invariant under the tangent lift of this action. Then so does$H$ and M are invariant under the cotangent lift of the action. Projections $\overline{M}$ and $\overline{H}$ on $\overline{T^*Q}$ are defined to be

$$\overline{M} = \overline{\mathbb{F}L}\left(\overline{N}\right), \qquad \overline{H} = \overline{E}_L \circ \overline{\mathbb{F}L}^{-1},$$

respectively. Here, $\overline{\mathbb{F}L}$ is the projection of $\mathbb{F}L$ and it is from the reduced tangent bundle $\overline{TQ}$ to the reduced cotangent bundle $\overline{T^*Q}$. Fix the following notation

$$\overline{\sigma} = \overline{\mathbb{F}L} \circ \overline{X}, \quad \overline{\Gamma}_{H,M} = \overline{\mathbb{F}L}_* \overline{\Gamma}_{L,N}. \tag{3.15}$$

Now, using the section $\overline{\sigma}$ of $\overline{\pi}_Q : \overline{T^*Q} \to \overline{Q}$ such that $\overline{\sigma}\left(\overline{Q}\right) \subseteq \overline{M}$, define a vector field on $\overline{Q}$ as follows:

$$\overline{\Gamma}_{H,M}^{\overline{\sigma}} = T\overline{\pi}_Q \circ \overline{\Gamma}_{H,M} \circ \overline{\sigma}. \tag{3.16}$$

We can now state the following theorem.

Theorem 13. *Let $\overline{\sigma}$ be a one-form on Q such that $\overline{\sigma}\left(\overline{Q}\right) \subset \overline{M}$ and $\left(\overline{\sigma} \circ \overline{\tau}_Q\right)^* \overline{\omega}_Q \in \mathscr{I}\left(\overline{F}_L^o\right)$. Then the following conditions are equivalent:*

(*i*) *$\overline{\Gamma}_{H,M}^{\overline{\sigma}}$ and $\overline{\Gamma}_{H,M}$ are $\overline{\sigma}$-related;*

(*ii*) *$d\left(\overline{H} \circ \left(\overline{\sigma} \circ \overline{\tau}_Q\right)\right) \in \overline{F}_L^o$.*

Definition 5. A section $\overline{\sigma}$ of $\overline{\pi}_Q$ satisfying conditions of Theorem 13 will be called a solution of the reduced constrained Hamilton – Jacobi problem given by the constraint manifold $M \subseteq T^*Q$, the Hamiltonian H and the action of the group G on Q.

Again, the space of G-invariant one-forms σ on Q are in one-to-one correspondence to sections $\overline{\sigma}$ of $\overline{\pi}_Q$.

Proposition 4. *Let σ be a G-invariant one-form on Q. Then σ is a solution of the constrained Hamilton – Jacobi problem if and only if $\overline{\sigma}$ is a solution of the reduced constrained Hamilton – Jacobi problem.*

Corollary 4. *Let $\overline{\sigma}$ be a section of $\overline{\pi}_Q$ in the condition of Theorem* 13 *satisfying $T\overline{\sigma}\left(T\overline{Q}\right) \subseteq \overline{F}$. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}^{\overline{\sigma}}_{H,M}$ *and* $\overline{\Gamma}_{H,M}$ *are* $\overline{\sigma}$*-related;*

(*ii*) $d\left(\overline{H} \circ \overline{\sigma}\right) = 0$.

Here, F is the transformed distribution of F_L on T^*Q via $\mathbb{F}L$ and $\overline{F}$ is the projection on $\overline{T^*Q}$ of F. Finally, we can give a Hamiltonian version of Theorem 12.

Theorem 14. *Let $\overline{\sigma}$ be a section of $\overline{\pi}_Q$ in the condition of Theorem* 13 *such that $\overline{F}_L$ is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}^{\overline{\sigma}}_{H,M}$ *and* $\overline{\Gamma}_{H,M}$ *are* $\overline{\sigma}$*-related;*

(*ii*) $d\left(\overline{H} \circ \overline{\sigma}\right) = 0$.

4. CLASSIFICATION OF NONHOLONOMIC SYSTEMS UNDER SYMMETRY

Depending on the relative position of the distribution F_L associated with the external forces, and the vertical space associated with the projection ρ_{TQ}, we can define three different types of nonholonomic systems. In the following classification, $(\mathscr{V}_N)_v$ is the space of vertical vectors at v in N with respect to the projection ρ_N (see (3.1)), and $\mathscr{H}_v$ is the vector space defined in Eq. (2.13).

1) Pure kinematic case: $(\mathscr{V}_N)_v \cap \mathscr{H}_v = 0$ and $T_vN = (\mathscr{V}_N)_v + \mathscr{H}_v$ for all $v \in N$.

2) Horizontal symmetries: $(\mathscr{V}_N)_v \cap \mathscr{H}_v = (\mathscr{V}_N)_v$ for all $v \in N$ which is equivalent to $(\mathscr{V}_N)_v \subset \mathscr{H}_v$ for all $v \in N$.

3) General case: $0 \subsetneq (\mathscr{V}_N)_v \cap \mathscr{H}_v \subsetneq (\mathscr{V}_N)_v$ for all $v \in N$.

For a detailed development of the general case, see [34].

The nonholonomic momentum map. We present a generalization of the momentum map for nonholonomic systems with symmetry obtained by Bloch et al [3] and extended by M. de León et al [28]. This generalization considers the union of subspaces in $\mathfrak{g}$ in order to account for the constrained dynamics to a submanifold N. In the case of the tangent space, for every $q \in Q$ we define the following vector subspace of $\mathfrak{g}$:

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} \ : \ \xi_{TQ}(v_q) \in (F_L)_{v_q}, \forall v_q \in T_qQ \cap N\}. \tag{4.1}$$

We consider now the disjoint union of these vector spaces, taken at all points of Q, i. e.,

$$\mathfrak{g}^N = \bigsqcup_{q \in Q} \mathfrak{g}^q \subseteq Q \times \mathfrak{g} \tag{4.2}$$

and let $\pi : \mathfrak{g}^N \to Q$ be a canonical projection. Notice that this is not a vector bundle, because $\mathfrak{g}^N$ is not a differentiable manifold. We will understand this as a generalized vector bundle, which induces another generalized vector bundle $(\mathfrak{g}^N)^*$ defining its fibers as the dual vector spaces of the fibers of $\mathfrak{g}^N$. So, we define the nonholonomic momentum map as $J_L^{nh} : TQ \to (\mathfrak{g}^N)^*$ by

$$J_L^{nh}(v_q)\,(\xi) = \alpha_L(\xi_{TQ})(v_q), \quad \text{for all} \quad v_q \in TQ, \quad \xi \in \mathfrak{g}^N. \tag{4.3}$$

Let ξ be a section of $\mathfrak{g}^N$. Then we can define a map $J_{L,\xi}^{nh} : TQ \to \mathbb{R}$ by putting

$$J_{L,\xi}^{nh}\,(v_q) = J_L^{nh}(v_q)\big(\xi\,(q)\,\big),\ \forall v_q \in TQ.$$

Furthermore, ξ induces a vector field Θ on Q as follows:

$$\Theta(q) = \big(\xi(q)\big)_Q(q), \tag{4.4}$$

through which we can write a Noether theorem, see [28].

Theorem 15.

$$\Gamma_{L,N}(J_{L,\xi}^{nh}) = \Theta^c(L), \tag{4.5}$$

where Θ^c denotes the complete lifting of Θ onto a vector field on TQ.

If it happens that ξ is a section of $\mathfrak{g}^N \to Q$ that is constant, i. e., $\xi(q) = \xi \in \mathfrak{g}$ for all q, then $\Theta^c = \xi_{TQ}$, which in view of the G-invariance of L implies that $\Gamma_{L,N}(J_{L,\xi}^{nh}) = 0$.

4.1. Hamilton – Jacobi Theory in the Pure Kinematic Case

In the pure kinematic case, we have the following direct sum decomposition of the tangent bundle of the constraint submanifold:

$$TN = \mathscr{V}_N \oplus \mathscr{H}. \tag{4.6}$$

It is important to remark that in this case the nonholonomic momentum map is trivial, i. e., $J^{nh} \equiv 0$. It is satisfied that $\mathscr{U} = \mathscr{H}$ and, hence, $\mathscr{U}$ is just the horizontal distribution of a principal connection in ρ_N. This implies that the projected distribution $\overline{\mathscr{U}}$ is equal to the tangent bundle $T\overline{N}$. Therefore, $\overline{\omega}_L$ is an almost symplectic two-form on $\overline{N}$. To work with a symplectic two-form on $\overline{N}$, we define the one-form on N,

$$\tilde{\alpha} = \iota_{\Gamma_{L,N}}\big(\mathbf{h}^* d\,(j^*\alpha_L) - d\mathbf{h}^*\,(j^*\alpha_L)\,\big), \tag{4.7}$$

where $j : N \hookrightarrow TQ$ is the inclusion map and $\mathbf{h}$ is horizontal projection of $\mathscr{H}$. Here $\tilde{\alpha}$ is G-invariant and horizontal, hence, it projects to a one-form $\overline{\alpha}$ on $\overline{N}$. In this situation, the reduced equations of motion are given in the following theorem.

Theorem 16. *The reduced equation of motion can be written as follows:*

$$\iota_{\overline{\Gamma}_{L,N}}\overline{\omega} = d\overline{E}_L - \overline{\alpha}, \tag{4.8}$$

where $\overline{\omega} = d\overline{\alpha}_L$ is a symplectic two-form and $\overline{\alpha}_L$ is the projection on $\overline{N}$ of $\boldsymbol{h}^*\,(j^*\alpha_L)$. Moreover, we can prove that $\iota_{\overline{\Gamma}_{L,N}}\overline{\alpha} = 0$.

Let $\overline{X}$ be a section of the tangent bundle of $\overline{\tau}_Q$ such that $\overline{X}\,\big(\overline{Q}\big)$ takes values in $\overline{N}$. Notice that $\overline{\Gamma}_{L,N}$ is a vector field on the reduced manifold $\overline{N}$, so that we define a vector field $\overline{\Gamma}_{L,N}^{\overline{X}}$ on $\overline{Q}$.

Theorem 17. *Let $\overline{X}$ be a section of $\overline{\tau}_Q$ such that $\overline{X}\,\big(\overline{Q}\big) \subseteq \overline{M}$ and $\overline{X}^*\overline{\omega} = 0$. Then the following two conditions are equivalent:*

(*i*) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(*ii*) $d(\overline{E}_L \circ \overline{X}) = -\overline{X}^*\overline{\alpha}$.

Proof. By using Eq. (4.8) we have that

$$\overline{X}^{*}\left(\iota_{\overline{\Gamma}_{L,N}}\overline{\omega}\right)=d\left(\overline{E}_{L}\circ\overline{X}\right)-\overline{X}^{*}\overline{\alpha}.$$

So, condition (ii) can be rewritten as follows:

$$\overline{X}^{*}\left(\iota_{\overline{\Gamma}_{L,N}}\overline{\omega}\right)=0, \tag{4.9}$$

i. e., $\overline{X}\left(\overline{Q}\right)$ is an isotropic submanifold of $\overline{N}$. On the other hand, in the purely kinematic case, the dimension of G is equal to the maximum number of independent constraints. Then

$$\dim\left(\overline{N}\right)=2\big(\dim\left(Q\right)-\dim\left(G\right)\big)=\frac{1}{2}\big(\overline{X}\left(\overline{Q}\right)\big).$$

Therefore, $\overline{X}\left(\overline{Q}\right)$ is a Lagrangian submanifold of $\overline{N}$. Now suppose that (i) holds. Then by Eq. (4.8) we have that

$$\overline{X}^{*}\left(\iota_{\overline{\Gamma}_{L,N}}\overline{\omega}\right)=\iota_{\overline{\Gamma}_{L,N}^{\overline{X}}}\left(\overline{X}^{*}\overline{\omega}\right)=0.$$

Conversely, we only need to use the fact that $\overline{X}\left(\overline{Q}\right)$ is a Lagrangian submanifold of $\overline{N}$ to prove (ii). □

Remark 1. An interesting but very particular example is given by a section $\overline{X}$ of $\overline{\tau}_{Q}$ which satisfies the conditions of Theorem 17 and Theorem 7. Then $\overline{\Gamma}_{L,N}^{\overline{X}}$ and $\overline{\Gamma}_{L,N}$ are $\overline{X}$-related if and only if $\overline{X}^{*}\overline{\alpha}=0$.

Legendre transformation. Let us introduce the following reduced objects:

$$\overline{\sigma}=\overline{\mathbb{F}L}\circ\overline{X},\qquad\overline{\theta}=\overline{\mathbb{F}L}^{-1^{*}}\overline{\alpha}\qquad\overline{\Omega}=\overline{\mathbb{F}L}^{-1^{*}}\overline{\omega}.$$

Note that $\overline{\sigma}$ is a section of $\overline{\pi}_{Q}$ and we have that $\overline{\sigma}^{*}\overline{\theta}=\overline{X}^{*}\overline{\alpha}$. Furthermore, the vector field $\overline{\Gamma}_{H,M}$ is the solution of the reduced equation

$$\iota_{\overline{\Gamma}_{H,M}}\overline{\Omega}=d\overline{H}-\overline{\theta}, \tag{4.10}$$

So, we can now state the following theorem.

Theorem 18. *Let $\overline{\sigma}$ be a section of $\overline{\pi}_{Q}$ such that $\overline{\sigma}\left(\overline{Q}\right)\subseteq\overline{M}$ and $\overline{\sigma}^{*}\overline{\Omega}=0$. Then the following conditions are equivalent:*

(i) *$\overline{\Gamma}_{H,M}^{\overline{\sigma}}$ and $\overline{\Gamma}_{H,M}$ are $\overline{\sigma}$-related;*

(ii) *$d\left(\overline{H}\circ\overline{\sigma}\right)=\overline{\sigma}^{*}\overline{\theta}$.*

Linear constraints. Let X be a G-invariant vector field on Q such that $X\left(Q\right)\subseteq N_{\ell}$ and $\mathbb{F}L\circ X$ is a closed one form. Then there exists a unique section $\overline{X}$ of $\overline{\tau}_{Q}$ which satisfies that

$$\rho_{TQ}\circ X=\overline{X}\circ\rho_{Q}.$$

Hence, $\overline{X}$ satisfies that $\overline{X}\left(\overline{Q}\right)\subseteq\overline{N}_{\ell}$. On the other hand, by the condition $X\left(Q\right)\subseteq N_{\ell}$ we have that for all $v_{q}\in N_{\ell}\cap T_{q}Q$

$$T_{q}X\left(v_{q}\right)\in T_{X(q)}N_{\ell}.$$

Then, taking into account the linearity of N_{ℓ}, for each a,

$$S^{*}\left(d\psi^{a}{}_{|X(q)}\right)\left(T_{q}X\left(v_{q}\right)\right)=\left(\psi^{a}\left(X\left(q\right)+tv_{q}\right)\right)'=0,$$

that is, for all $v_q \in N_\ell \cap T_qQ$,

$$T_qX\,(v_q) \in (F_L)_{X(q)} \Leftrightarrow T_qX\,(v_q) \in \mathscr{H}_{X(q)}.$$

Therefore, we find that $\mathbf{h}\big(T_qX\,(v_q)\big) = T_qX\,(v_q)$. With this, for all $v_q \in T_qQ$

$$\begin{aligned}\Big[X^*\mathbf{h}^*\left(j^*\alpha_L\right)(q)\Big]\,(v_q) &= \alpha_L\big(X\,(q)\big)\Big(T_{X(q)}j\big(\mathbf{h}\left(T_qX\,(v_q)\right)\big)\Big)\\ &= \alpha_L\big(X\,(q)\big)\Big(T_{X(q)}j\big(T_qX\,(v_q)\big)\Big)\\ &= \alpha_L\big(X\,(q)\big)\big(T_qX\,(v_q)\big)\\ &= \Big[X^*\left(\alpha_L\right)(q)\Big]\,(v_q)\,,\end{aligned}$$

that is,

$$X^*\mathbf{h}^*\left(j^*\alpha_L\right) = X^*\left(\alpha_L\right). \tag{4.11}$$

Notice that, by an abuse of notation, we have denoted X and $j \circ X$ (i. e., the restriction of the codomain to N) by the same notation X. Then, again by some abuse of notation, we have that

$$\overline{X}^*\overline{\omega} = d\overline{X}^*\overline{\alpha}_L = d\overline{X^*\mathbf{h}^*\left(j^*\alpha_L\right)} = d\overline{X^*\alpha_L}. \tag{4.12}$$

On the other hand, we know that $\mathbb{F}L \circ X$ is a closed one-form equivalent to $X^*\omega_L = dX^*\alpha_L = 0$, and hence, $\overline{X}^*\overline{\omega} = 0$. Using this fact, we can give the following two equivalent corollaries:

Corollary 5. *Let X be a G-invariant vector field on Q, and X a section such that $X\,(Q) \subseteq N$ and $\mathbb{F}L \circ X$ is a closed one-form σ. Then the following conditions are equivalent:*

(i) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(ii) $d\left(\overline{E}_L \circ \overline{X}\right) = \overline{X}^*\overline{\alpha}$.

Corollary 6. *Let σ be a G-invariant closed one-form on Q. Then the following conditions are equivalent:*

(i) $\overline{\Gamma}_{LH,M}^{\overline{\sigma}}$ *and* $\overline{\Gamma}_{H,M}$ *are* $\overline{\sigma}$*-related;*

(ii) $d\left(\overline{H} \circ \overline{\sigma}\right) = \overline{\sigma}^*\overline{\theta}$.

Notice that we are projecting G-invariant one-forms σ on Q to get one-forms $\overline{\sigma}$ on $\overline{Q}$ as we made with the vector fields.

Chaplygin systems. We will say that our nonholonomic system is a Chaplygin system when the constraint manifold N is given by the total space of a horizontal distribution $\mathscr{C}$ of the principal connection ρ_Q. Then $TQ = \mathscr{C} \oplus \mathscr{V}_Q$, where $\mathscr{V}_Q$ is the vertical distribution of ρ_Q. Its associated horizontal lift $\mathbf{h}_{\mathscr{C}}$ is a section of $T\rho_Q : TQ \to T\overline{Q}$ that satisfies

$$\mathbf{h}_{\mathscr{C}}\left(v_{\overline{q}}\right) \in \mathscr{C}_q,\ \forall v_{\overline{q}} \in T_{\overline{q}}\overline{Q}.$$

Then the restriction of $T\rho_Q$ to $\mathscr{C}$ is an isomorphism and, therefore, $\mathscr{C} \cong \overline{\mathscr{C}} \cong T\overline{Q}$. So, we may define a Lagrangian function $L^* : T\overline{Q} \to \mathbb{R}$ such that

$$L^*\left(v_{\overline{q}}\right) = L\big(\mathbf{h}_{\mathscr{C}}\left(v_{\overline{q}}\right)\big),\ \forall v_{\overline{q}} \in T_{\overline{q}}\overline{Q}.$$

A computation shows that $\overline{\omega} = \omega_{L^*}$ and $\overline{E}_L = E_{L^*}$. Thus, the reduced equation of motion is

$$\iota_{\overline{\Gamma}_{L,N}}\omega_{L^*} = dE_{L^*} - \overline{\alpha}. \tag{4.13}$$

The proof of the following assertion can be found in [14].

Theorem 19. *Let $\overline{X}$ be vector field of $\overline{Q}$ such that $\mathbb{F}L^* \circ \overline{X}$ is closed. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}_{L,N}^{\overline{X}}$ *and* $\overline{\Gamma}_{L,N}$ *are* $\overline{X}$*-related;*

(*ii*) $d\left(E_{L^*} \circ \overline{X}\right) = \overline{X}^* \overline{\alpha}$.

Notice that to consider, for instance, $\overline{X}^* \overline{\alpha}$ we should see α as a form on $T\overline{Q}$.

Theorem 20. *Let $\overline{\sigma}$ be a closed one form on $\overline{Q}$. Then the following conditions are equivalent:*

(*i*) $\overline{\Gamma}_{H,M}^{\overline{\sigma}}$ *and* $\overline{\Gamma}_{H,M}$ *are* $\overline{\sigma}$*-related;*

(*ii*) $d\left(\overline{H} \circ \overline{\sigma}\right) = \overline{\sigma}^* \overline{\theta}$.

4.2. Hamilton – Jacobi Theory for the Horizontal Case

In the horizontal symmetry case, one has $\mathscr{V}_N \cap \mathscr{H} = \mathscr{V}_N$. So that $\mathscr{V}_N$ becomes a subdistribution of $\mathscr{H}$. The importance of this case lies in the fact that the nonholonomic momentum map turns out to be the classical momentum map $J_L : TQ \to \mathfrak{g}^*$. In fact, $\mathfrak{g}^q = \mathfrak{g}$ for all q in Q. So, by using the Marsden – Weinstein reduction [35], for each value $\mu \in \mathfrak{g}^*$ (assuming that all the values are regular) we have a reduced symplectic manifold

$$\left(\mathscr{P}_\mu = J_L^{-1}(\mu)/G_\mu, \omega_{L,\mu}\right),$$

where G_μ is the isotropy group of μ for the coadjoint action and $\omega_{L,\mu}$ is the unique symplectic two-form on $\mathscr{P}_\mu$ satisfying

$$\rho_{L,\mu}^* \omega_{L,\mu} = i_{L,\mu}^* \omega_L.$$

Here, $\rho_{L,\mu}$ is the canonical projection from $J_L^{-1}(\mu)$ to the reduced manifold $\mathscr{P}_\mu$, whereas $i_{L,\mu}$ is the inclusion map from $J_L^{-1}(\mu)$ to the tangent bundle TQ. Let us consider the quotient $N_\mu = N/G_\mu$ and the projection

$$\rho_{TQ,\mu} : TQ \to (TQ)_\mu = TQ/G_\mu.$$

We assume that N and $J_L^{-1}(\mu)$ have a clean intersection N'. Notice that N' is obviously G_μ-invariant and, hence, it projects onto a submanifold N_μ of $\mathscr{P}_\mu$. Furthermore, the distribution F_L induces a distribution F_L' on $J_L^{-1}(\mu)$ along N' given by the intersection of F_L with the tangent space of $J_L^{-1}(\mu)$ at each vector of N'. Again, the G_μ-invariance of F_L' projects onto a distribution $F_{L,\mu}$ on $\mathscr{P}_\mu$ along N_μ. Now we are ready to state the following theorem [28].

Theorem 21. *The projection $(\Gamma_{L,N})_\mu$ of the restriction of $\Gamma_{L,N}$ to N' is a solution of the reduced equations of motion*

$$\iota_{(\Gamma_{L,N})_\mu} \omega_{L,\mu} - d(E_L)_\mu \in F_{L,\mu}^o; \qquad (\Gamma_{L,N})_\mu \in TN_\mu, \tag{4.14}$$

where $(E_L)_\mu$ is the reduced energy.

We have the following equality:

$$\rho_{TQ,\mu}^* \left(TN_\mu^o\right) = TN^o. \tag{4.15}$$

Let us study the codistribution $F_{L,\mu}^o$. Notice that, for each $U_{v_q} \in T_{v_q}(TQ)$ and $g \in G_\mu$, we have

$$S\big(T_{v_q}\phi_g^T\left(U_{v_q}\right)\big) = \left(\phi_g^T(v_q) + t\big[T_{v_q}\left(\tau_Q \circ \phi_g^T\right)\left(U_{v_q}\right)\big]\right)'_{|0}$$

$$
\begin{aligned}
&= \left(T_q\phi_g\left(v_q\right) + t\left[T_{v_q}\left(\phi_g \circ \tau_Q\right)\left(U_{v_q}\right)\right]\right)'_{|0} \\
&= \left(T_q\phi_g\left(v_q + t\left[T_{v_q}\tau_Q\left(U_{v_q}\right)\right]\right)\right)'_{|0} \\
&= \left[T_{v_q}\phi_g^T \circ S\right]\left(U_{v_q}\right).
\end{aligned}
$$

Therefore, the vertical endomorphism S, defined in (2.2), is G_μ-invariant and it can be projected onto a map S_μ defined on $T\left(TQ\right)_\mu$ so that $T\rho_{TQ,\mu} \circ S = S_\mu \circ T\rho_{TQ,\mu}$. Therefore, we write

$$\rho^*_{TQ,\mu} S^*_\mu = S^* \rho^*_{TQ,\mu}. \tag{4.16}$$

Thus, by definition, we have that

$$\rho^*_{L,\mu} F^o_{L,\mu} = i^*_{L,\mu}\big(S^*\left(TN^o\right)\big).$$

Then, taking into account Eq. (4.15) and Eq. (4.16), one writes

$$\rho^*_{L,\mu} F^o_{L,\mu} = \left(\rho_{TQ,\mu} \circ i_{L,\mu}\right)^*\left(S^*_\mu\left(TN^o_\mu\right)\right) = \left(j_{L,\mu} \circ \rho_{L,\mu}\right)^*\left(S^*_\mu\left(TN^o_\mu\right)\right),$$

where $j_{L,\mu} : \mathscr{P}_\mu \hookrightarrow \left(TQ\right)_\mu$. Then one arrives at

$$F^o_{L,\mu} = j^*_{L,\mu}\left(S^*_\mu\left(TN^o_\mu\right)\right). \tag{4.17}$$

Let us consider the constraints ψ^a which characterize (locally) N. Then, using Eq. (4.17), there exists a family of (local) one-forms

$$j^*_{L,\mu} S^*_\mu\left(\overline{d\psi^a}\right) \tag{4.18}$$

on $\left(TQ\right)_\mu$, generating $F^o_{L,\mu}$.

Hamilton – Jacobi analysis. Let τ_Q be the tangent bundle projection, then we define the following projection:

$$\tau_{Q,\mu} : \left(TQ\right)_\mu \to Q_\mu = Q/G_\mu, \qquad \tau_{Q,\mu} \circ j_{L,\mu} = \tau_\mu : \mathscr{P}_\mu \to Q_\mu.$$

Any section X_μ of τ_μ such that $X_\mu\left(Q_\mu\right) \subseteq N_\mu$ induces a vector field $\left(\Gamma_{L,N}\right)^X_\mu$ on Q_μ by projecting $\left(\Gamma_{L,N}\right)_\mu$ according to the following diagram.

$$
\begin{array}{ccc}
N_\mu & \xrightarrow{\left(\Gamma_{L,N}\right)_\mu} & TN_\mu \\
{\scriptstyle X_\mu}\uparrow & & \downarrow{\scriptstyle T\left(\tau_\mu|_{N_\mu}\right)} \\
Q_\mu & \xrightarrow{\left(\Gamma_{L,N}\right)^X_\mu} & TQ_\mu
\end{array}
\tag{4.19}
$$

We can now consider an ideal $\mathscr{I}\left(F^o_{L,\mu}\right)$ in the algebra of forms on $\mathscr{P}_\mu$ in the form (see the generating forms given in 4.18)

$$\beta_i \wedge j^*_{L,\mu} S^*_\mu\left(\overline{d\psi^a}\right),\ \beta_i \in \Lambda^k\left(\mathscr{P}_\mu\right).$$

Theorem 22. *Let X_μ be a section of τ_μ such that $X_\mu\left(Q_\mu\right) \subseteq N_\mu$ and $\left(X_\mu \circ \tau_\mu\right)^* \omega_{L,\mu} \in \mathscr{I}\left(F^o_{L,\mu}\right)$. Then the following conditions are equivalent:*

(i) $\left(\Gamma_{L,N}\right)^X_\mu$ *and* $\left(\Gamma_{L,N}\right)_\mu$ *are X_μ-related;*

(ii) $d\Big(\left(E_L\right)_\mu \circ \left(X_\mu \circ \tau_\mu\right)\Big) \in F^o_{L,\mu}$.

Proof Notice that the condition $(X_\mu \circ \tau_\mu)^* \omega_{L,\mu} \in \mathscr{I}(F^o_{L,\mu})$ implies that

$$\iota_{(\Gamma_{L,N})_\mu} (X_\mu \circ \tau_\mu)^* \omega_{L,\mu} \in F^o_{L,\mu} \tag{4.20}$$

along X_μ. Now, let us prove that

$$(X_\mu \circ \tau_\mu)^* F^o_{L,\mu} = F^o_{L,\mu} \tag{4.21}$$

along X_μ. There exists a vector field X on Q such that

$$X_\mu \circ \rho_{Q,\mu} = \rho_{L,\mu} \circ X,$$

where $\rho_{Q,\mu} : Q \to Q_\mu$ is the canonical projection. Then, for each a, using Eq. (4.16), we have that

$$\begin{aligned}
\rho^*_{L,\mu}\left[(X_\mu \circ \tau_\mu)^* j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right)\right] &= (X_\mu \circ \tau_\mu \circ \rho_{L,\mu})^* j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right) \\
&= [X \circ \tau_Q \circ i_{L,\mu}]^* \rho^*_{L,\mu}\left(j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right)\right) \\
&= [X \circ \tau_Q \circ i_{L,\mu}]^* \left((j_{L,\mu} \circ \rho_{L,\mu})^* S^*_\mu \left(\overline{d\psi^a}\right)\right) \\
&= [X \circ \tau_Q \circ i_{L,\mu}]^* \left((\rho_{TQ,\mu} \circ i_{L,\mu})^* S^*_\mu \left(\overline{d\psi^a}\right)\right) \\
&= [i_{L,\mu} \circ X \circ \tau_Q \circ i_{L,\mu}]^* \left(\rho^*_{TQ,\mu} S^*_\mu \left(\overline{d\psi^a}\right)\right) \\
&= [i_{L,\mu} \circ X \circ \tau_Q \circ i_{L,\mu}]^* S^* \left(\rho^*_{TQ,\mu}\left(\overline{d\psi^a}\right)\right) \\
&= i^*_{L,\mu} S^* \left(\rho^*_{TQ,\mu}\left(\overline{d\psi^a}\right)\right) \\
&= \rho^*_{L,\mu}\left[j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right)\right].
\end{aligned}$$

Hence, for all a

$$(X_\mu \circ \tau_\mu)^* \left[j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right)\right] = j^*_{L,\mu} S^*_\mu \left(\overline{d\psi^a}\right).$$

Notice that we are using the fact that

$$[i_{L,\mu} \circ X \circ \tau_Q \circ i_{L,\mu}]^* S^* = i^*_{L,\mu} S^*,$$

which can be proved using coordinates. Therefore, Eq. (4.21) and Eq. (4.14) imply that

$$(X_\mu \circ \tau_\mu)^* [\iota_{(\Gamma_{L,N})_\mu} \omega_{L,\mu}] \in F^o_{L,\mu} \Leftrightarrow d\left((E_L)_\mu \circ (X_\mu \circ \tau_\mu)\right) \in F^o_{L,\mu}.$$

So, taking into account Eq. (4.20) we can easily prove that (i) implies (ii). Conversely, notice that

$$(\Gamma_{L,N})_\mu \circ X_\mu = T(X_\mu \circ \tau_\mu) \circ (\Gamma_{L,N})_\mu \circ X_\mu + V_\mu,$$

where $V_\mu \in Ker(T\tau_\mu)$ for each point. So, $V_\mu \in TN_\mu$. On the other hand, using the conditions $(X_\mu \circ \tau_\mu)^* \omega_{L,\mu} \in \mathscr{I}(F^o_{L,\mu})$ and $(X_\mu \circ \tau_\mu)^* [\iota_{(\Gamma_{L,N})_\mu} \omega_{L,\mu}] \in F^o_{L,\mu}$ we have that

$$\begin{aligned}
(X_\mu \circ \tau_\mu)^* [\iota_{V_\mu} \omega_{L,\mu}] &= (X_\mu \circ \tau_\mu)^* [\iota_{(\Gamma_{L,N})_\mu \circ X_\mu - T(X_\mu \circ \tau_\mu) \circ (\Gamma_{L,N})_\mu \circ X_\mu} \omega_{L,\mu}] \\
&= (X_\mu \circ \tau_\mu)^* [\iota_{(\Gamma_{L,N})_\mu \circ X_\mu} \omega_{L,\mu}] - (X_\mu \circ \tau_\mu)^* [\iota_{T(X_\mu \circ \tau_\mu) \circ (\Gamma_{L,N})_\mu \circ X_\mu} \omega_{L,\mu}] \\
&= (X_\mu \circ \tau_\mu)^* [\iota_{(\Gamma_{L,N})_\mu \circ X_\mu} \omega_{L,\mu}] - \iota_{(\Gamma_{L,N})_\mu \circ X_\mu} [(X_\mu \circ \tau_\mu)^* \omega_{L,\mu}] \in F^o_{L,\mu}.
\end{aligned}$$

Then

$$\omega_{L,\mu}\big(V_\mu, T(X_\mu \circ \tau_\mu)(Z)\big) = 0, \ \forall Z \in F_{L,\mu}. \tag{4.22}$$

Let us now consider $V \in F_{L,\mu} \cap Ker(T\tau_\mu)$. Hence, using the fact that $V_\mu \in Ker(T\tau_\mu)$, we obtain

$$\omega_{L,\mu}(V_\mu, V) = 0. \tag{4.23}$$

Let us prove this identity carefully. Consider $U, W \in T(TQ)$ such that

$$T\rho_{Q,\mu}\big(T\tau_Q(U)\big) = T\rho_{Q,\mu}\big(T\tau_Q(W)\big) = 0. \tag{4.24}$$

Then

$$\omega_L(U,W)=0. \tag{4.25}$$

In fact, let $\left(q^i,x^j\right)$ be a local system of coordinates on Q adapted to $\rho_{Q,\mu}$, i. e.,

$$\rho_{Q,\mu}\left(q^i,x^j\right)=q^i.$$

Then the family of vectors satisfying Eq. (4.24) is generated by the partial derivatives $\partial/\partial x^i$, $\partial/\partial\dot{x}^i$ and $\partial/\partial\dot{q}^j$. Next, as a result of the G-invariance,

$$dL\left(\frac{\partial}{\partial x^i}\right)=0\Rightarrow\frac{\partial L}{\partial x^i}=0,\ \forall i.$$

Then we only have to use the local expression of ω_L to prove Eq. (4.25) and, hence, Eq. (4.23). Notice that Eq. (4.23) is slightly more general than Lemma 1. Let $Z\in F_{L,\mu}$ be at a point along X_μ. Then

$$Z=T\left(X_\mu\circ\tau_\mu\right)(Z)+V,$$

with $V\in F_{L,\mu}\cap Ker\left(T\tau_\mu\right)$. Here we are using the fact that $T\left(X_\mu\circ\tau_\mu\right)$ preserves $F_{L,\mu}$ via Eq. (4.21). Then, taking into account Eq. (4.22) and Eq. (4.23), we obtain

$$\omega_{L,\mu}(V_\mu,Z)=0. \tag{4.26}$$

i. e., $V_\mu\in TN_\mu\cap F_{L,\mu}^{\perp}=\{0\}$ and, therefore,

$$\left(\Gamma_{L,N}\right)_\mu\circ X_\mu=T\left(X_\mu\circ\tau_\mu\right)\circ\left(\Gamma_{L,N}\right)_\mu\circ X_\mu,$$

which proves (i). Let us highlight a particular case. Suppose that

$$S(TN)\subseteq TN. \tag{4.27}$$

Then $F_L=TN$. Therefore, the identity $\mathscr{V}_N\cap\mathscr{H}=\mathscr{V}_N$ is trivially satisfied, i. e., this case is a horizontal symmetric case. Notice that Eq. (4.27) can be interpreted as a compatibility condition between the constraints and the vertical endomorphism. It is easy to check that $F_L=TN$ implies Eq. (4.27).

Definition 6. A section X_μ of τ_μ satisfying the conditions of Theorem 22 will be called a μ-solution of the reduced constrained Hamilton – Jacobi problem given by the constraint manifold $N\subseteq TQ$, the regular Lagrangian L and the action of the group G on Q.

It is remarkable that, as we have noticed in the proof of the previous theorem, there is a one-to-one correspondence between G_μ-invariant vector fields X on Q such that $X(Q)\subseteq J_L^{-1}(\mu)$ and sections X_μ of τ_μ. This correspondence will be denoted by $X\mapsto X_\mu$. The proof of this correspondence is analagous to the correspondence $X\mapsto\overline{X}$ (see (3.14)). Thus, it works as a reconstruction process from μ-solutions of the reduced constrained Hamilton – Jacobi problem to the solutions of the constrained Hamilton – Jacobi problem.

Proposition 5. *Let X be a G_μ-invariant vector field on Q such that $X(Q)\subseteq J_L^{-1}(\mu)$. Then X is a solution of the constrained Hamilton – Jacobi problem if and only if X_μ is a μ-solution of the reduced constrained Hamilton – Jacobi problem.*

Let us now study analogous consequences to Corollary 1 and Theorem 7.

Corollary 7. *Let X_μ be a section of τ_μ in the conditions of Theorem 22 such that $TX_\mu(TQ_\mu)\subseteq F_{L,\mu}$. Then the following conditions are equivalent:*

(*i*) $\left(\Gamma_{L,N}\right)_\mu^X$ *and* $\left(\Gamma_{L,N}\right)_\mu$ *are X_μ-related;*

(*ii*) $d\big((E_L)_\mu\circ X_\mu\big)=0$.

Proof. Notice that, because of the existence of the section X_μ, τ_μ is a submersion. Then the one-form $d\big((E_L)_\mu \circ X_\mu\big)$ is zero if and only if

$$d\big((E_L)_\mu \circ (X_\mu \circ \tau_\mu)\big) = \tau_\mu^*\Big[d\big((E_L)_\mu \circ X_\mu\big)\Big] = 0.$$

Finally, as an immediate consequence of the condition $TX_\mu(TQ_\mu) \subseteq F_{L,\mu}$, we have that the one-form $d\big((E_L)_\mu \circ (X_\mu \circ \tau_\mu)\big)$ is in the annihilator $F_{L,\mu}^o$ if and only if $d\big((E_L)_\mu \circ (X_\mu \circ \tau_\mu)\big) = 0$. □

Completely nonholonomic systems. We study now an application of the Chow – Rashevskii theorem (Theorem 2) to the present case. Notice, once again, that $F_{L,\mu}$ is not a distribution defined on the whole space $\mathscr{P}_\mu$ since it is only defined on N_μ. In order to overcome this, define distribution $G_{L,\mu}$ on $\mathscr{P}_\mu$ by restricting and taking the quotient of G_L (see Section 2). Obviously, $G_{L,\mu}$ is a restriction of $F_{L,\mu}$ over N_μ. Let us consider the family of (local) vector fields $\mathscr{S}_{N_\mu}$ tangent to $G_{L,\mu}$ whose domains have a nonempty intersection with N_μ. Then $F_{L,\mu}$ is the distribution generated by $\mathscr{S}_{N_\mu}$ along N_μ. Thus, $F_{L,\mu}$ is called completely nonholonomic if the reiterated Lie bracket of vector fields in $\mathscr{S}_{N_\mu}$ generates $T_{N_\mu}\mathscr{P}_\mu$. In this case we will say that

$$[F_{L,\mu}, F_{L,\mu}], [F_{L,\mu}, [F_{L,\mu}, F_{L,\mu}]], \dots$$

spans the vector bundle $T_{N_\mu}\mathscr{P}_\mu$. Obviously, $G_{L,\mu}$ is completely nonholonomic if and only if $F_{L,\mu}$ is completely nonholonomic. We are now ready to give a reduced version of Theorem 7.

Theorem 23. *Let X_μ be a section of τ_μ in the conditions of Theorem* 22 *such that $F_{L,\mu}$ is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $(\Gamma_{L,N})_\mu^X$ *and* $(\Gamma_{L,N})_\mu$ *are X_μ-related;*

(*ii*) $d\big((E_L)_\mu \circ X_\mu\big) = 0$.

Proof. Due to the fact that $G_{L,\mu}$ is completely nonholonomic we can use Proposition 1 to have that $d\big((E_L)_\mu \circ X_\mu\big) \in G_{L,\mu}$ if and only if $d\big((E_L)_\mu \circ X_\mu\big) = 0$. □

Proposition 6. *F_L' is completely nonholonomic if and only if $F_{L,\mu}$ is completely nonholonomic.*

To prove this result we only have to change the group G by the isotropy group G_μ in the proof of Proposition 3. Note that the condition of being completely nonholonomic over F_L' is generally not related to the same condition over F_L. This is a consequence of the fact that, in general, the momentum map J_L has no relation to the vertical endomorphism S and the constraint manifold N.

The Legendre transformation. Now let us study the Hamiltonian case via the Legendre transformation $\mathbb{F}L$. Consider the classical momentum map $J_H : T^*Q \to \mathfrak{g}^*$ of the cotangent bundle. Then $J_H = J_L \circ \mathbb{F}L^{-1}$. Again, we have a reduced symplectic manifold

$$\left(\mathscr{Q}_\mu = J_H^{-1}(\mu)/G_\mu, \omega_{Q,\mu}\right),$$

where G_μ is the isotropy group of μ for the coadjoint action and $\omega_{Q,\mu}$ is the unique symplectic two-form on $\mathscr{Q}_\mu$ such that

$$\rho_{H,\mu}^*\omega_{Q,\mu} = i_{H,\mu}^*\omega_Q,$$

where $\rho_{H,\mu} : J_H^{-1}(\mu) \to \mathscr{Q}_\mu$ is the canonical projection and $i_{H,\mu} : J_H^{-1}(\mu) \hookrightarrow T^*Q$ is the inclusion map. Notice that $\mathbb{F}L$ can be restricted to a map from $J_L^{-1}(\mu)$ to $J_H^{-1}(\mu)$. In fact, for each vector $v_q \in J_L^{-1}(\mu)$ we have

$$J_H\big(\mathbb{F}L(v_q)\big) = J_L(v_q) = \mu.$$

So, quotienting the restiction, we get a map $\mathbb{F}L_\mu$ from $\mathscr{Q}_\mu$ to $\mathscr{P}_\mu$ characterized by the following identity:

$$\mathbb{F}L_\mu \circ \rho_{L,\mu} = \rho_{H,\mu} \circ \mathbb{F}L_{|J_L^{-1}(\mu)}. \tag{4.28}$$

Denote the intersection of M and $J_H^{-1}(\mu)$ by M'. M' is obviously G_μ-invariant and, hence, it projects onto a submanifold M_μ of $\mathscr{Q}_\mu$. Notice that $M' = \mathbb{F}L(N')$ and, hence,

$$M_\mu = \mathbb{F}L_\mu(N_\mu).$$

Let us again fix some notations:

- $\pi_{Q,\mu} : (T^*Q)_\mu \to Q_\mu = Q/G_\mu$ is the projection of π_Q;
- $j_{H,\mu} \circ \pi_{Q,\mu} = \pi_\mu : \mathscr{Q}_\mu \to Q_\mu$, where $j_{H,\mu} : \mathscr{Q}_\mu \hookrightarrow (T^*Q)_\mu$ is the inclusion map.

Any section σ_μ of π_μ such that $\sigma_\mu(Q_\mu) \subseteq M_\mu$ induces a vector field $(\Gamma_{H,M})^\sigma_\mu$ on Q_μ by projecting $(\Gamma_{H,M})_\mu$ following the next diagram:

$$\begin{array}{ccc} M_\mu & \xrightarrow{(\Gamma_{H,M})_\mu} & TM_\mu \\ {\scriptstyle\sigma_\mu}\uparrow & & \downarrow{\scriptstyle T(\pi_\mu|_{M_\mu})} \\ Q_\mu & \xrightarrow{(\Gamma_{H,M})^\sigma_\mu} & TQ_\mu \end{array} \tag{4.29}$$

Theorem 24. *Let σ_μ be a section of π_μ such that $\sigma_\mu(Q_\mu) \subseteq M_\mu$ and $(\sigma_\mu \circ \tau_\mu)^* \omega_{Q,\mu} \in \mathscr{I}(F^o_{L,\mu})$. Then the following conditions are equivalent:*

(i) $(\Gamma_{H,M})^\sigma_\mu$ and $(\Gamma_{H,M})_\mu$ are σ_μ-related;

(ii) $d\big(H_\mu \circ (\sigma_\mu \circ \tau_\mu)\big) \in F^o_{L,\mu}$.

We will now give a Hamiltonian version of Corollary 7 and Theorem 23.

Theorem 25. *Let σ_μ be a one-form on Q such that $\sigma_\mu(\overline{Q}) \subset \overline{M}$ and $(\sigma_\mu \circ \overline{\tau}_Q)^* \overline{\omega}_Q \in \mathscr{I}(\overline{F}^o_L)$. Then the following conditions are equivalent:*

(i) $\overline{\Gamma}^{\sigma_\mu}_{H,M}$ and $\overline{\Gamma}_{H,M}$ are σ_μ-related;

(ii) $d\big(\overline{H} \circ (\sigma_\mu \circ \overline{\tau}_Q)\big) \in \overline{F}^o_L$.

Definition 7. A section σ_μ of π_μ satisfying the conditions of Theorem 25 will be called a μ-solution of the reduced constrained Hamilton – Jacobi problem given by the constraint manifold $M \subseteq T^*Q$, the Hamiltonian H and the action of the group G on Q.

Once again, the space of G_μ-invariant one-forms σ on Q such that $\sigma(Q) \subseteq J_H^{-1}(\mu)$ are in one-to-one correspondence with sections σ_μ of π_μ

Proposition 7. *Let σ be a G_μ-invariant one-form on Q such that $\sigma(Q) \subseteq J_H^{-1}(\mu)$. Then σ is a solution of the constrained Hamilton – Jacobi problem if and only if σ_μ is a μ-solution of the reduced constrained Hamilton – Jacobi problem.*

Corollary 8. *Let σ_μ be a section of π_μ in the condition of Theorem 25 satisfying $T\sigma_\mu(TQ_\mu) \subseteq F_\mu$. Then the following conditions are equivalent:*

(*i*) $(\Gamma_{H,N})^{\sigma}_{\mu}$ *and* $(\Gamma_{H,N})_{\mu}$ *are* σ_μ*-related;*

(*ii*) $d(H_\mu \circ \sigma_\mu) = 0$

Here, F_μ is the transformed distribution of F_L via $\overline{\mathbb{F}L}$. Finally, we can give a Hamiltonian version of Theorem 25.

Theorem 26. *Let* σ_μ *be a section of* π_μ *in the condition of Theorem* 25 *such that* $F_{L,\mu}$ *is completely nonholonomic. Then the following conditions are equivalent:*

(*i*) $(\Gamma_{H,N})^{\sigma}_{\mu}$ *and* $(\Gamma_{H,N})_{\mu}$ *are* σ_μ*-related;*

(*ii*) $d(H_\mu \circ \sigma_\mu) = 0$.

As a particular case, let us consider again the case in which N_ℓ is a vector subbundle of TQ and the constraint functions ψ^a are linear, i. e., ψ^a induce one-forms $\overline{\psi}^a$ on Q which generate N^o_ℓ. Observe that the following diagram is commutative.

$$\begin{array}{ccc} N_\ell & \xrightarrow{F^o_L} & T^*(TQ) \\ \downarrow{\scriptstyle \tau_Q} & & \uparrow{\scriptstyle T^*\tau_Q} \\ Q & \xrightarrow{N^o_\ell} & T^*Q \end{array} \tag{4.30}$$

i. e., $\tau^*_Q(N^o_\ell) = F^o_L$. Let us prove, in this case, the Hamiltonian version of the corollary.

Corollary 9. *Let* σ *be a* G_μ*-invariant one-form on* Q *such that* $\sigma(Q) \subseteq N_\ell$ *and* $d\sigma \in \mathscr{I}(N^o_\ell)$. *Let* σ_μ *be the projection of* σ *as a section of* π_μ. *Then the following conditions are equivalent:*

(*i*) $(\Gamma_{H,N})^{\sigma}_{\mu}$ *and* $(\Gamma_{H,N})_{\mu}$ *are* σ_μ*-related;*

(*ii*) $d\big(H_\mu \circ (\sigma_\mu \circ \tau_\mu)\big) \in F^o_{L,\mu}$.

Proof. Let σ be a G_μ-invariant one-form on Q such that $\sigma(Q) \subseteq N_\ell$ and $d\sigma \in \mathscr{I}(N^o_\ell)$. The condition $d\sigma \in \mathscr{I}(N^o_\ell)$ implies that

$$\tau^*_Q[d\sigma] = d\left(\tau^*_Q\sigma\right) = (\sigma \circ \tau_Q)^* \omega_Q \in \mathscr{I}(F^o_L).$$

This is a consequence of the commutativity of diagram (4.30). Consider the projection σ_μ of σ as a section of π_μ. Then, with some slight abuse of notation, we have

$$\begin{aligned} \rho^*_{L,\mu}\left[(\sigma_\mu \circ \tau_\mu)^* \omega_{Q,\mu}\right] &= (\sigma_\mu \circ \tau_\mu \circ \rho_{L,\mu})^* \omega_{Q,\mu} = \left(\sigma_\mu \circ \rho_{Q,\mu} \circ \tau_{Q|J_L^{-1}(\mu)}\right)^* \omega_{Q,\mu} \\ &= \left(\rho_{L,\mu} \circ \sigma \circ \tau_{Q|J_L^{-1}(\mu)}\right)^* \omega_{Q,\mu} = \left(\sigma \circ \tau_{Q|J_L^{-1}(\mu)}\right)^* \left[\rho^*_{L,\mu}\omega_{Q,\mu}\right] \\ &= \left(\sigma \circ \tau_{Q|J_L^{-1}(\mu)}\right)^* \left[i^*_{L,\mu}\omega_Q\right] = i^*_{L,\mu}\left[(\sigma \circ \tau_Q)^* \omega_Q\right] \in i^*_{L,\mu}\mathscr{I}(F^o_L). \end{aligned}$$

Notice that $F^o_{L,\mu}$ is characterized by the condition

$$\rho^*_{L,\mu}F^o_{L,\mu} = i^*_{L,\mu}F^o_L,$$

and, therefore, $(\sigma_\mu \circ \tau_\mu)^* \omega_{Q,\mu} \in \mathscr{I}\left(F^o_{L,\mu}\right)$, i. e., we are in the conditions of Theorem 24. Let us study the second condition (*ii*), which is equivalent to

$$d(H_\mu \circ (\sigma_\mu \circ \tau_\mu) \circ \rho_{L,\mu}) \in i^*_{L,\mu}F^o_L.$$

Observe that

$$\begin{aligned} d\left(H_\mu \circ (\sigma_\mu \circ \tau_\mu) \circ \rho_{L,\mu}\right) &= d(H_\mu \circ \sigma_\mu \circ \tau_{Q,\mu} \circ i_{L,\mu}) = d(H_\mu \circ \sigma_\mu \circ \rho_{Q,\mu} \circ \tau_{Q|J_L^{-1}(\mu)}) \\ &= d(H_\mu \circ \rho_\mu \circ \sigma \circ \tau_{Q|J_L^{-1}(\mu)}) = d(H \circ \sigma \circ \tau_{Q|J_L^{-1}(\mu)}) \\ &= (\tau_Q \circ i_{L,\mu})^* \left[d\left(H \circ \sigma\right)\right] = i_{L,\mu}^* \left[\tau_Q^* d\left(H \circ \sigma\right)\right]. \end{aligned}$$

Therefore, $d\left(H \circ \sigma\right) \in N_\ell^o$ implies condition (ii) of the above corollary. □

Finally, we have the following corollary for the Lagrangian counterpart.

Corollary 10. *Let X be a G_μ-invariant vector field on Q such that $X\left(Q\right) \subseteq N$ and that $d\left(\mathbb{F}L \circ X\right) \in \mathscr{I}\left(N_\ell^o\right)$. Let X_μ be the projection of X as a section of τ_μ. Then the following conditions are equivalent:*

(i) $\left(\Gamma_{L,N}\right)_\mu^X$ *ad* $\left(\Gamma_{L,N}\right)_\mu$ *are X_μ-related;*

(ii) $d\left(\left(E_L\right)_\mu \circ \left(X_\mu \circ \tau_\mu\right)\right) \in F_{L,\mu}^o$.

5. EXAMPLES

5.1. Free Particle with a Nonholonomic Constraint

In this section, we will exhibit Hamilton – Jacobi formulations of a very well-known example of nonholonomic systems, namely, a free particle under a nonholomic constraint [4]. We cite [36] for the Hamilton – Jacobi theory of this system in the realm of Poisson framework. In this paper, we discuss the Hamilton – Jacobi formalism of the system under the symmetry reduction. In this case, we consider a three-dimensional space $Q = \mathbb{R}^3$ with coordinates (x, y, z). The tangent bundle is $TQ = \mathbb{R}^3 \times \mathbb{R}^3$ with coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$. The Lagrangian function is the kinetic energy of a single particle with mass m, that is,

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

whereas the nonholonomic submanifold is defined to be

$$N_\ell = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in TQ : \psi = \dot{z} - y\dot{x} = 0\}.$$

Using the linear constraint function ψ, we introduce the differential form $\bar{\psi} = dz - ydx$. This implies that N_ℓ can be defined as the image space of two linear independent vector fields, that is,

$$N_\ell = im\{\xi_1 = \partial_x + y\partial_z, \xi_2 = \partial_y\}.$$

The Jacobi Lie bracket $[\xi_2, \xi_1]$ of these vector fields is another linear independent vector field $\xi_3 = \partial_z$. This gives that N_ℓ is a completely nonholonomic constraint submanifold so that we are in the realm of Theorem (3). The distribution F_L is computed as

$$F_L = span\{\partial_x + y\partial_z, \partial_y, \partial_{\dot{x}}, \partial_{\dot{y}}, \partial_{\dot{z}}\}, \tag{5.1}$$

so that the subbundle in (2.13) turns out to be

$$\mathscr{H} = TN_\ell \cap F_L = span\{\partial_x + y\partial_z, \partial_y + \dot{x}\partial_{\dot{z}}, \partial_{\dot{x}} + y\partial_{\dot{z}}, \partial_{\dot{y}}\}. \tag{5.2}$$

The Hamilton – Jacobi formalism for this picture can be obtained by the Hamilton – Jacobi theorem (3). For this case, we search for a solution X of the relation

$$d(E_L \circ X) = 0. \tag{5.3}$$

In order to deal with this identity directly, we shall apply the following reduction procedure to the system.

Translation in the xz-plane in Q under the group action of $G = \mathbb{R}^2$ is given by

$$G \times Q : \left(\left(r, s\right), \left(x, y, z\right)\right) \to \left(x + r, y, z + s\right).$$

This results in a quotient manifold $\overline{Q} \simeq \mathbb{R}$ with coordinate (y), and the reduced tangent bundle is $\overline{TQ} \simeq \mathbb{R}^4$, which is covered by coordinates $(y, \dot{x}, \dot{y}, \dot{z})$. The vertical subbundle $\mathscr{V}$ is spanned by the pair of vector fields ∂_x, and ∂_y. The intersection of $\mathscr{V}$ and the distribution $\mathscr{H}$ in (5.2) satisfies the following strict inclusions:

$$0 \subsetneq \mathscr{V} \cap \mathscr{H} = span\{\partial_x + y\partial_z\} \subsetneq \mathscr{V},$$

which says that we are in the general case. Note that the reduction of the tangent bundle projection is

$$\overline{\tau}_Q : \overline{TQ} \simeq \mathbb{R}^4 \to \overline{Q} \simeq \mathbb{R} : (y, \dot{x}, \dot{y}, \dot{z}) \to y,$$

so that a section $\overline{X}$ of this projection is a function

$$\overline{X} : \overline{Q} \simeq \mathbb{R} \to \overline{TQ} \simeq \mathbb{R}^4 : y \to \left(y, \overline{A}(y), \overline{B}(y), \overline{C}(y)\right),$$

where $\overline{A}$, $\overline{B}$ and $\overline{C}$ are arbitrary functions depending on the free variable y. Reduction of the distribution F_L in (5.1) under the group action results in

$$\overline{F}_L = span\{\partial_y, \partial_{\dot{x}}, \partial_{\dot{y}}, \partial_{\dot{z}}\} = T\overline{TQ}.$$

This implies that the condition $T\overline{X}\left(T\overline{Q}\right) \subset \overline{F}_L$, presented in Corollary 3, is trivially satisfied. We can use this corollary in order to exhibit the Hamilton – Jacobi formalism of the reduced system. The reduced constraint submanifold $\overline{N_\ell}$ is 3-dimensional with coordinates $(y, \dot{x}, \dot{y})$ where $\dot{z} = y\dot{x}$. Therefore, if we insist that $\overline{X}\left(\overline{Q}\right) \subset \overline{N_\ell}$, we take that $\overline{C} = y\overline{A}$. The reduced energy $\overline{E}_L$ is computed to be

$$\overline{E}_L : \overline{TQ} \to \mathbb{R} : (y, \dot{x}, \dot{y}, \dot{z}) \to \frac{1}{2}m\left(\left(1 + y^2\right)\dot{x}^2 + \dot{y}^2\right).$$

We write the second condition $d(\overline{E}_L \circ \overline{X}) = 0$ in Corollary 3 for the present case, and obtain

$$\frac{1}{2}m\left(\left(1 + y^2\right)\overline{A}(y)^2 + \overline{B}(y)^2\right) = c.$$

A solution of this equation can be given by $\overline{A}(y) = c_1/\sqrt{1 + y^2}$ and $\overline{B}(y) = c_2$, so that the reduced section is

$$\overline{X}(y) = \left(y, \frac{c_1}{\sqrt{1 + y^2}}, c_2, \frac{c_1 y}{\sqrt{1 + y^2}}\right).$$

Let us now construct a vector field X according to the one-to-one correspondence in (3.14). An easy computation shows that the dynamics generated by X is

$$\dot{x} = c_1/\sqrt{1 + y^2}, \qquad \dot{y} = c_2/\sqrt{1 + y^2}, \qquad \dot{z} = c_1 y/\sqrt{1 + y^2}. \tag{5.4}$$

Note that X solves the Hamilton – Jacobi problem in the unreduced picture (5.3). Compare the system in (5.4) with the one in [36].

5.2. The Two-wheeled Carriage

In this subsection, we shall write the Hamilton – Jacobi formalisms both of reduced and unreduced formulations of a two-wheeled carriage [6, 19]. The nonholonomic character of this example was investigated in [14]. Later, it was discussed in the framework of geometrization of generalized Chaplygin systems in [15]. In a more recent work [21], the Hamilton – Jacobi formalism of this example was studied in the framework of linear almost Poisson structures. In the present work, in addition to the previous studies, we shall investigate the Hamilton – Jacobi formalism of the reduced nonholonomic system. To this end, we start by introducing the basic ingredients of the system.

The configuration space Q of the two-wheeled carriage is the product manifold $SE(2) \times \mathbb{T}^2$ where $SE(2)$ is the special Euclidean group of $\mathbb{R}^2$, physically representing the rigid body motion

in the xy-plane, with local coordinates (x, y, θ), whereas $\mathbb{T}^2$ is a two-torus, realizing the left and the right wheels of the carriage, with local coordinates (ϕ_1, ϕ_2). For the fiber coordinates of the tangent bundle TQ, we consider the induced coordinates $(\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2)$. The Lagrangian function generating the equation of motion is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + m_0 l\dot{\theta}\left(\dot{y}\cos\theta - \dot{x}\sin\theta\right) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}C\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right),$$

where m_0 is the mass of the body without wheels, m is the mass of the total system, J is the moment of inertia when it rotates as a whole about the vertical axis passing through the point (x, y), C is the axial moment of inertia, and l is the distance of the point (x, y) to the center of mass. In order to impose no lateral sliding and no sliding on both wheels, we introduce the following set of linear nonholonomic constraints:

$$\begin{aligned}
\psi^1 &= \dot{x} + \frac{a\cos\theta}{2}\dot{\phi}_1 + \frac{a\cos\theta}{2}\dot{\phi}_2,\\
\psi^2 &= \dot{y} + \frac{a\sin\theta}{2}\dot{\phi}_1 + \frac{a\sin\theta}{2}\dot{\phi}_2,\\
\psi^3 &= \dot{\theta} + \frac{a}{2r}\dot{\phi}_1 - \frac{a}{2r}\dot{\phi}_2,
\end{aligned}$$

where a is the radius of the wheels and r is half the length of the axle. These constraint functions determine a 7-dimensional constraint submanifold

$$N_\ell = \{v \in TQ : \psi^a(v) = 0, \qquad a = 1, 2, 3\}$$

equipped with a coordinate chart $(x, y, \theta, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$.

Let us analyze the nonholonomic character of the submanifold N_ℓ. Now it is straightforward to define a set of constraint one-forms

$$\begin{aligned}
\bar{\psi}^1 &= dx + \frac{a\cos\theta}{2}d\phi_1 + \frac{a\cos\theta}{2}d\phi_2,\\
\bar{\psi}^2 &= dy + \frac{a\sin\theta}{2}d\phi_1 + \frac{a\sin\theta}{2}d\phi_2,\\
\bar{\psi}^3 &= d\theta + \frac{a}{2r}d\phi_1 - \frac{a}{2r}d\phi_2
\end{aligned} \tag{5.5}$$

spanning the codistribution N_ℓ^o. From this, compute that the constraint submanifold N_ℓ is the image space of two vector fields

$$N_\ell = im\{\xi_1, \xi_2\}$$

where

$$\begin{aligned}
\xi_1 &= -\frac{a\cos\theta}{2}\partial_x - \frac{a\sin\theta}{2}\partial_y - \frac{a}{2r}\partial_\theta + \partial_{\phi_1},\\
\xi_2 &= -\frac{a\cos\theta}{2}\partial_x - \frac{a\sin\theta}{2}\partial_y + \frac{a}{2r}\partial_\theta + \partial_{\phi_2}.
\end{aligned}$$

The Jacobi – Lie bracket of these two vector fields is computed and labeled as ξ_3 as follows:

$$\xi_3 = [\xi_1, \xi_2] = -\frac{a^2\sin\theta}{2r^2}\partial_x + \frac{a^2\cos\theta}{2r^2}\partial_y.$$

We compute the Jacobi – Lie bracket of ξ_1 and ξ_3 and thus arrive at

$$\xi_4 = [\xi_1, \xi_3] = \frac{a^3\cos\theta}{4r^2}\partial_x + \frac{a^3\sin\theta}{4r^2}\partial_y.$$

These four vector fields, ξ_1, ξ_2, ξ_3 and ξ_4, are linearly independent. The dimension of the tangent space T_qQ at each q in Q is five. In order to find a fifth linear independent vector field, we compute the following Jacobi – Lie brackets:

$$[\xi_1, \xi_4] = -\frac{a^2}{4r}\xi_3, \quad [\xi_2, \xi_3] = -\xi_4, \quad \text{and} \quad [\xi_3, \xi_4] = 0.$$

It is easy now to observe that such an iteration will not produce a fifth linearly independent vector field. This says that the constraint submanifold N_ℓ is not completely nonholonomic. In this case, the Hamilton – Jacobi problem of this system is solved by Theorem 1. This reads

$$d\left(E_L \circ X\right) \in N_\ell^o. \tag{5.6}$$

Instead of finding a solution X to this problem, we now employ reduction to the system and then lift solutions of the Hamilton – Jacobi problem of the reduced system.

Consider a Lie group action $G = SE\left(2\right)$ on $Q = SE\left(2\right) \times \mathbb{T}^2$ by simple group action of G on to the first factor $SE\left(2\right)$ in Q while acting trivially on $\mathbb{T}^2$. Notice that the two-wheeled carriage is a Chaplygin system. In this case, the reduced dynamics can be written as [37]

$$\iota_{\overline{\Gamma_{L,N_\ell}}}\omega_{\overline{L}} = dE_{\overline{L}} + \overline{\alpha_{L,N_\ell}},$$

where the reduced Lagrangian is computed to be

$$\overline{L}\left(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2\right) = \frac{1}{8}ma^2\left(\dot{\phi}_1 + \dot{\phi}_2\right)^2 + \frac{Ja^2}{8r^2}\left(\dot{\phi}_2 - \dot{\phi}_1\right)^2 + \frac{1}{2}C\left(\dot{\phi}_1^2 + \dot{\phi}_2^2\right),$$

whereas the one-form $\overline{\alpha_{L,N_\ell}}$ is

$$\overline{\alpha_{L,N_\ell}} = \frac{a^3 m_o l}{4r^2}\left(\dot{\phi}_2 - \dot{\phi}_1\right)\dot{\phi}_2 d\phi_1 - \frac{a^3 m_o l}{4r^2}\left(\dot{\phi}_1 - \dot{\phi}_2\right)\dot{\phi}_1 d\phi_2.$$

So that we apply the Hamilton – Jacobi theorem (19) to this case, and arrive at the following Hamilton – Jacobi equation:

$$d\left(E_{\overline{L}} \circ \overline{X}\right) = -\overline{X}^* \overline{\alpha_{L,N_\ell}}.$$

The energy function $E_{\overline{L}}$ corresponding to the reduced Lagrangian $\overline{L}$ is equal to $\overline{L}$. A straightforward calculation proves that two solutions of this equation are

$$\overline{X}_1 = e^{\frac{K}{R}\phi_2}\frac{\partial}{\partial\phi_1}, \quad \overline{X}_2 = e^{\frac{K}{R}\phi_1}\frac{\partial}{\partial\phi_2},$$

where $K = -m_o l a^3/4r^2$ and $R = \frac{1}{4}ma^2 + Ja^2/4r^2 + C$. We compute the horizontal lifts of these solutions of the reduced equation to the manifold Q by means of a connection [14]

$$\left(\partial_{\phi_1}\right)^h = \partial_{\phi_1} - a\cos\theta\partial_x - a\sin\theta\partial_y - \frac{a}{r}\partial_\theta$$
$$\left(\partial_{\phi_2}\right)^h = \partial_{\phi_2} - a\cos\theta\partial_x - a\sin\theta\partial_y + \frac{a}{r}\partial_\theta,$$

so that we have two solutions of the unreduced Hamilton – Jacobi problem (5.6)

$$X_1 = e^{\frac{K}{R}\phi_2}\left(\partial_{\phi_1} - a\cos\theta\partial_x - a\sin\theta\partial_y - \frac{a}{r}\partial_\theta\right)$$
$$X_2 = e^{\frac{K}{R}\phi_1}\left(\partial_{\phi_1} - a\cos\theta\partial_x - a\sin\theta\partial_y - \frac{a}{r}\partial_\theta\right).$$

5.3. Appell Particle

In this subsection, we present a nonholonomic system, the so-called Appell particle, which is equipped with a nonlinear constraint function [18, 38]. Let us start first by introducing its Lagrangian framework. Take $Q = \mathbb{R}^3$ with coordinates (x, y, z) and $TQ = \mathbb{R}^3 \times \mathbb{R}^3$ with fiber coordinates $(\dot{x}, \dot{y}, \dot{z})$. Consider the following Lagrangian function:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz, \tag{5.7}$$

where m is a constant realizing the mass, and g is the gravitational acceleration. The corresponding energy function E_L, generated by this Lagrangian functions, is obtained as

$$E_L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz. \tag{5.8}$$

The Lagrangian function given in (5.7) is regular, so that the Poincaré-Cartan two-form becomes symplectic and has the following explicit expression:

$$\omega_L = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}).$$

Consider now, in addition to the Lagrangian L, a constraint submanifold

$$N = \{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \in TQ : \Phi = \dot{z} - a(\dot{x}^2 + \dot{y}^2)^{1/2} = 0\} \tag{5.9}$$

defined in terms of the nonlinear constraint function $\Phi = \dot{z} - a(\dot{x}^2 + \dot{y}^2)^{1/2}$, where a is a real constant. The constraint is ideal since $\dot{x}\Phi_{\dot{x}} + \dot{y}\Phi_{\dot{y}} + \dot{z}\Phi_{\dot{z}} = 0$ on the constraint submanifold N. Here, the subscript notation in $\Phi_{\dot{x}^i}$ shows the partial derivative of Φ with respect to $\dot{x}^i$. The tangent bundle and its annihilator are computed to be

$$TN^0 = span\left\{d\dot{z} - a\frac{\dot{x}d\dot{x} + \dot{y}d\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right\}, TN = span\left\{\partial_x, \partial_y, \partial_z, \partial_{\dot{z}} + \frac{\dot{x}\partial_{\dot{x}} + \dot{y}\partial_{\dot{y}}}{a\sqrt{\dot{x}^2 + \dot{y}^2}}, \dot{y}\partial_{\dot{x}} - \dot{x}\partial_{\dot{y}}\right\}$$

provided that $\dot{x}^2 + \dot{y}^2$ is nonzero. The bundle of reaction forces is

$$F_L^0 = S^*(TN^0) = span\left\{dz - a\frac{\dot{x}dx + \dot{y}dy}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right\}.$$

In this case, the distribution F_L turns out to be

$$F_L = span\left\{\partial_z + \frac{\dot{x}\partial_x + \dot{y}\partial_y}{a\sqrt{\dot{x}^2 + \dot{y}^2}}, \dot{y}\partial_x - \dot{x}\partial_y, \partial_{\dot{x}}, \partial_{\dot{y}}, \partial_{\dot{z}}\right\}.$$

Notice that we have

$$\mathscr{H} = TN \cap F_L = \left\{\partial_z + \frac{\dot{x}\partial_x + \dot{y}\partial_y}{a\sqrt{\dot{x}^2 + \dot{y}^2}}, \dot{y}\partial_x - \dot{x}\partial_y, \dot{y}\partial_{\dot{x}} - \dot{x}\partial_{\dot{y}}, \partial_{\dot{z}} + \frac{\dot{x}\partial_{\dot{x}} + \dot{y}\partial_{\dot{y}}}{a\sqrt{\dot{x}^2 + \dot{y}^2}}\right\}.$$

In order to arrive at the Hamilton – Jacobi formalism of this system, one needs to directly employ Theorem 6. Instead of assuming that the solutions of the Hamilton – Jacobi problem satisfy $TX(TQ) \subseteq F_L$, we can employ Corollary 1 as well. Further, we see from the Jacobi – Lie bracket

$$[\dot{y}\partial_x - \dot{x}\partial_y, \partial_{\dot{x}}] = \partial_y$$

that F_L is completely nonholonomic. This implies that for the Hamilton – Jacobi formalism we can also use Theorem 7. Let us record here the substitution of the constraint (5.9) into the Lagrangian function L in (5.7). This results in the following Lagrangian and energy functions:

$$L^* = \frac{1}{2}m\big((1 + a^2)\dot{x}^2 + (1 + a^2)\dot{y}^2\big) - mgz, \qquad E_L^* = \frac{1}{2}m\big((1 + a^2)\dot{x}^2 + (1 + a^2)\dot{y}^2\big) + mgz.$$

In this case, the pullback of the symplectic two-form ω_L to the constraint submanifold N implies that

$$\omega_N = mdx \wedge d\dot{x} + mdy \wedge d\dot{y} + \frac{am\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}dz \wedge d\dot{x} + \frac{am\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}dz \wedge d\dot{y}.$$

We consider the following symmetry:

$$G = \left\{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in \mathbb{R}\right\}$$

of the Appell particle. The action of this group to $Q = \mathbb{R}^3$ is simply by the action of the matrix to the column vector $[x \quad y \quad z]^T$. Thus, the reduced manifold can be written as

$$\overline{Q} = Q/G = \{(r, z) : r \in \mathbb{R}^{>0}, z \in \mathbb{R}\}.$$

The action of the group on the base manifold is linear, so that its lift is the same with the action. This implies that the reduced tangent manifold can be written as

$$\overline{TQ} = TQ/G = \{(r, z, \dot{r}, \dot{z}) : r \in \mathbb{R}^{>0}, z \in \mathbb{R}, \dot{r} \in \mathbb{R}^{>0}, \dot{z} \in \mathbb{R}\}.$$

We identify this reduced manifold with the tangent bundle $T\overline{Q}$ of the reduced manifold $\overline{Q}$. In this reduced picture, the Lagrangian function (5.7) and the energy function (5.8) turn out to be

$$\overline{L} = \frac{1}{2}m(\dot{r}^2 + \dot{z}^2) - mgz, \qquad \overline{E_L} = E_{\overline{L}} = \frac{1}{2}m(\dot{r}^2 + \dot{z}^2) + mgz,$$

whereas the reduced symplectic two-form is computed to be

$$\overline{\omega_L} = mdr \wedge d\dot{r} + mdz \wedge d\dot{z}.$$

By the identification of $\overline{TQ}$ with $T\overline{Q}$, we arrive at the symplectic manifold $T^*\overline{Q}$ equipped with the canonical symplectic two-form $\omega_{\overline{Q}}$. Using the Legendre transformation $\mathbb{F}\overline{L}$, we pull $\omega_{\overline{Q}}$ back to $T\overline{Q}$. Notice that this pullback two-form is symplectic since $\overline{L}$ is regular and $\mathbb{F}\overline{L}^*\omega_{\overline{Q}}$ coincides with $\overline{\omega_L}$. On the other hand, the reduced constraint submanifold can be written as

$$\overline{N} = \{(r, z, \dot{r}, \dot{z}) \in \overline{TQ} : \dot{z} - a\dot{r} = 0\}.$$

Notice that the reduced system is a nonholonomic system with a linear constraint $\dot{z} - v\dot{r} = 0$, and that the two-tuple $(\overline{L}, \overline{N})$ determines a nonholonomic Lagrangian system on $T\overline{Q}$. Substituting the constraint $\dot{z} - v\dot{r} = 0$ into the Lagrangian results in the following Lagrangian:

$$\overline{L}^* = \frac{1}{2}m((1 + a^2)\dot{r}^2) - mgz, \qquad \overline{E_L}^* = \frac{1}{2}m((1 + a^2)\dot{r}^2) + mgz.$$

Restriction of the symplectic structure $\omega_{\overline{L}}$ to the constraint submanifold $\overline{N}$ gives

$$\omega_{\overline{N}} = mdx \wedge d\dot{r} + madz \wedge d\dot{r}.$$

Note that the tangent bundle of the reduced constraint submanifold $\overline{N}$ and its annihilator are computed to be

$$T\overline{N}^0 = span\{d\dot{z} - ad\dot{r}\}, \qquad T\overline{N} = span\{\partial_r, \partial_z, a\partial_{\dot{z}} + \partial_{\dot{r}}\},$$

whereas the reaction forces and its annihilator are computed to be

$$\overline{F}_L^0 = \overline{S}^*(T\overline{N}^0) = span\{dz - adr\}, \qquad \overline{F}_L = span\{a\partial_z + \partial_r, \partial_{\dot{r}}, \partial_{\dot{z}}\}$$

and the intersection is

$$T\overline{N} \cap \overline{F}_L^0 = span\{a\partial_z + \partial_r, a\partial_{\dot{z}} + \partial_{\dot{r}}\}.$$

Let us write the Hamilton – Jacobi formalism of the reduced nonholonomic dynamics $(\overline{L}, \overline{N})$. For a vector field $\overline{X} = A(r, z)\partial_r + B(r, z)\partial_z$ on $\overline{Q}$, we first require that $d(\mathbb{F}L \circ \overline{X}) \in \mathscr{I}(\overline{N}^0)$. Then the Hamilton – Jacobi equation is $d(\overline{E_L} \circ \overline{X}) \in \overline{N}^0$. This can explicitly be computed as

$$a\frac{1}{2}\frac{\partial}{\partial r}(A^2 + B^2) - \frac{1}{2}\frac{\partial}{\partial z}(A^2 + B^2) = g, \tag{5.10}$$

where g is the gravitational acceleration. Let us define a function $C(r, z) = A^2(r, z) + B^2(r, z)$. Then Eq. (5.10) can be written as

$$aC_r - C_z = 2g. \tag{5.11}$$

Then one reads that $C(r, z) = f(r + az) + 2gz$ is a solution of (5.11) for an arbitrary function f. So that, by choosing $f(r + az) - 2gz$ to be positive, we find coefficient functions A and B as follows:

$$A(r, z) = \sqrt{f(r + az) - 2gz}\cos\varepsilon, \qquad B(r, z) = \sqrt{f(r + az) - 2gz}\sin\varepsilon, \qquad \forall\varepsilon \in \mathbb{R}.$$

This results in a solution of the Hamilton – Jacobi equation (5.10) in the form

$$\overline{X} = \sqrt{f(r + az) - 2gz}\cos\varepsilon\partial r + \sqrt{f(r + az) - 2gz}\sin\varepsilon\partial z.$$

Consider the following lift from $\mathfrak{X}(\overline{Q})$ to $\mathfrak{X}(Q)$ given by

$$(\partial r)^h = \frac{x}{\sqrt{x^2+y^2}}\partial x + \frac{y}{\sqrt{x^2+y^2}}\partial y.$$

This enables us to arrive at a set of solutions

$$X = \sqrt{f(\sqrt{x^2+y^2}+az) - 2gz}\cos\varepsilon\left(\frac{x}{\sqrt{x^2+y^2}}\partial x + \frac{y}{\sqrt{x^2+y^2}}\partial y\right) + \sqrt{f(\sqrt{x^2+y^2}+az) - 2gz}\sin\varepsilon\partial z$$

of the Hamilton – Jacobi problem of the nonholonomic Lagrangian system (L, N). Here, f is an arbitrary function.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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